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Issues on Fuzzy EQ-logics and Their Algebraic Semantics

A Thesis submitted in partial fulfillment of the requirements of the degree of M.Sc. in Basic Engineering Sciences in Engineering Mathematics

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Benha 2020

ABSTRACT

A formal theory of new class of many-valued logics, called EQ-logics, has been recently introduced by M. Dyba and V. Novak. They are based on a special algebra of truth values called EQ-algebra introduced by V. Novak and open the door to an alternative development of mathematical fuzzy logics by starting with equivalence instead of implication. This direction can be considered as a generalization of the equational classical logics due to Gries and Schneider and it is justified by the idea presented by G.W.Leibniz that "a fully satisfactory logical calculus must be an equational one". Moreover, the formal proofs can be more effectively formed in an equational style; that is substitution of equals for equals, this makes it easier to discover proofs than it is when using the Hilbert style of deduction, rendering proofs more natural and more calculational.

This work continues the research in EQ-logics and their algebraic semantics that can be taken as special kind of fuzzy logics where completeness with respect to chains is the constitutive feature of all fuzzy logics. In particular, we introduce and study a class of separated (not necessarily good) lattice EQ-algebras that may be represented as subalgebras of products of linearly ordered ones. Such algebras are called representable. Namely, we enrich separated lattice EQ-algebras with a unary operation (the so called Baaz delta), fulfilling some additional assumptions. The resulting algebras are called $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras. One of the main results of this thesis is to characterize the class of representable $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras. We also provide a number of useful results, leading to this characterization. This also allows us to develop a more general fuzzy EQ-logic in which the basic connective is fuzzy equality and the implication is derived from the latter. Precisely, we formulate the corresponding $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-logic which is rich enough to enjoy the completeness

property and its set of truth values is formed by $\ell\mathrm{EQ}_{\boldsymbol{\Delta}}^{\mathrm{s}}$-algebras in which the fuzzy equality is one of the basic operations. The implication operation (as well as the corresponding connective) is derived. We in detail introduce syntax and semantics of the $\ell\mathrm{EQ}_{\boldsymbol{\Delta}}^{\mathrm{s}}$-logic and prove various theorems characterizing its properties including completeness. Formal proofs in this thesis proceed mostly in an equational style.

ACKNOWLEDGMENT

I would like to express my gratitude, respect and great appreciation to my supervisor, Associate Prof. Dr. Moataz El-Zekey, for his guidance, support, and kindness. He has always been available to aid me in one way or another when difficulties seem insurmountable. It is such a great privilege to work under his supervision and to have him as my "academic father", who allowed me much freedom in defining the overall scope of this work, and yet, at the same time offered continued support, ideas and encouragement.

The same gratitude and appreciation is also due to my co-supervisor, Dr. Monir Abdelaal, for his support and kindness. His unique, cheerful way of teaching and conducting research has always been a source of inspiration.

Many useful and stimulating discussions with my advisors played a role in the final development of the thesis and to them I owe my appreciation.

Finally, to my family, I owe great thanks, for their help in typing part of the manuscript and particularly for their support and understanding.

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March 2020

PUBLICATIONS

[1] M. El-Zekey, Mahmoud khalaf, On a representable class of separated lattice EQ-algebras, Journal of Basic and Environmental Sciences 7(1) (2020) 45-50.

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Chapter 1
Introduction

Mathematical logic has been for many years developed on the basis of implication as the main connective. In the recent past, new direction of the development has been initiated which is called equational logic [14, 29]. This logic is based on equality as the main connective. This direction is justified by the idea presented by G.W.Leibniz [2] that a fully satisfactory logical calculus must be an equational one. It is also argued by its proponents see [14] that equational logic is the pedagogically proper setting to do proofs because its main tool, substitution of equals for equals, makes it easier to discover proofs (than it is when using the Hilbert style of deduction), rendering proofs more natural and more calculational.

It brought an idea to develop also (fuzzy) many-valued logics on the basis of fuzzy equality (equivalence) as the principal connective. Accordingly, a formal theory of new different many-valued logics, called EQ-logics, has been recently introduced by M. Dyba and V. Novák [6]. They are based on a special algebra of truth values called EQ-algebra introduced by V. Novak in [22] (also [7, 8, 23]). Unlike the residuated lattices, the basic operation in it is a fuzzy equality while implication is derived from it. Its axioms reflect basic properties which fuzzy equality should have to fit the supporting structure, namely the ordered set. Its original motivation comes from the study of higher-order fuzzy logic [20] that was obtained as a generalization of simple type theory in the style of L. Henkin who developed in [16] a very elegant theory [1] in which the basic connective is equality.

As we believe that completeness w.r.t. chains is the constitutive feature of all fuzzy logics (see papers [3, 4] where reasons for this belief are presented), EQ-logics satisfying chain completeness are called here fuzzy EQ-logics.

Analysis of necessary properties of the fuzzy equality revealed that we cannot consider the fuzzy equality in full generality without means enabling us to deal with the classical (crisp) equality. This is possible using the Delta-connective. Thus, unlike the residuated fuzzy logics [10, 17] where the Δ-connective is interesting but dispensable option, the role of it in fuzzy equality-based logics is much deeper [5, 6]. We conclude that the general fuzzy equivalence is not sufficient and a crisp equivalence is necessary for well-behaving logic. On the other hand, the current investigation of fuzzy EQ-logics [5, 6] shows that goodness, is sufficient for the resulting logic has many reasonable properties including completeness and Delta-deduction theorem. The goodness axiom means that each element x is equal to $\mathbf{1}$ in the degree x. It implies that the algebra is separated (i.e., two elements equal in the degree $\mathbf{1}$ must be identical) but not vice-versa. Therefore, Separateness turned out to be indispensable for any kind of fuzzy equality based logic.

In this work, we continue developing the formal theory of fuzzy EQ-logics and their algebraic semantics. Namely, we focus more closely on the important role played by expanding the EQ-logics by the Delta-connective in our further development of both separated EQ-algebras and the corresponding EQ-logics. The long term goal of the research is to develop more general fuzzy EQ-logics whose semantics based on separated (need not to be good) EQ-algebras.

One of the important algebraic consequences of goodness axiom is axiomatizing the class of representable good EQ-algebras (expanded by Delta-connective) [7, 8]. This is mainly based on the fact that good EQ-algebras give raise to BCK-algebras [11, 25]. Further, development of this direction could also deal with the more challenging problem of characterizing separated (not necessarily good) representable EQ-algebras. This also allows us to develop a more general fuzzy EQ-logics whose semantics based on separated (need not to be good) EQ-algebras.

The thesis is made up of six chapters organized as follow:

In chapter 2: A summary of syntax and semantics of propositional logic are introduced. Moreover, all basic definitions and notions of formula, logical axioms, inference rules and formal proof are presented. While we also present short notes on soundness, and completeness of propositional logic [29].

In chapter 3: This chapter is divided into two parts; the first part is customized mainly for recalling the definitions of residuated lattices and BL algebra. The concept of EQ-algebras are introduced, the basic definitions, important essential properties, special kinds of EQ-algebras, and some examples of EQ-algebras [8, 23] are provided. Moreover, we display prelinear EQ-algebras, and also, we introduce the prefilters and filters of EQ-algebras [7]. Finally, we present characterizing both of the representable class of good EQ-algebras, good EQ-algebras with a unary operation "Δ", and its prelinear version [5, 7, 8]. The second part is dedicated for introducing an overview for the basic EQ-logic and show its fundamental properties whose good EQ-algebras as the algebraic structure of its truth values. Also, the completeness theorem of the basic EQ-logic is introduced [6]. As well as the prelinear EQ_{Δ}-logic and its completeness theorem are showed [5]. To this point, we discuss the previous studies that were introduced in the last years.

In chapter 4: We introduce and discuss a special type of EQ-algebras called $\ell\mathrm{EQ}_{\Delta}^{s}$-algebras. As well as, introducing and studying in-depth the filters and the congruences of $\ell\mathrm{EQ}_{\Delta}^{s}$-algebras. Moreover, characterizing the representable class of $\ell\mathrm{EQ}_{\Delta}^{s}$-algebras will be introduced.

In chapter 5: We present the $\ell\mathrm{EQ}_{\boldsymbol{\Delta}}^{s}$-logic and prove its main properties including the completeness theorem and the deduction theorem. It should be given emphasis to that formal proofs in this thesis proceed mostly in the equational style.

In chapter 6: The future work and conclusions obtained from the thesis are given.

Chapter 2
Equational Propositional Logic

Mathematical logic, or as we will simply say, "logic", represents the most general means of mathematical reasoning used by mathematicians and computers. Its core consists of the study of the form, meaning, use, and limitations of logical deductions, the so-called proofs.

Classical logic is usually presented as implication is the basic connective but there exists also approach based on equivalence as basic connective instead of implication which, however, gains gradually still more and more interest, too (see, e.g. [29]). There are at least two main reasons for that. First, equality (equivalence) seems to be more essential connective than implication. This direction is justified by the idea presented by G.W.Leibniz (cf. [2]) that a fully satisfactory logical calculus must be an equational one. Moreover, the formal proofs can be more effectively formed in an equational style. The second reason is also argued by its proponents (see, for example, [14]) that equational logic is the pedagogically proper setting to do proofs because its main tool, substitution of equals for equals, makes it easier to discover proofs (than it is when using the Hilbert style of deduction), rendering proofs more natural and more calculational. Both approaches are equivalent.

More precisely, in this chapter we introduce an overview of the simplest part of mathematical logic, the equational propositional logic, or simply equational logic (also namely, Boolean logic, propositional calculus, sentential logic, and sentential calculus). You will get acquainted with the notions of formula, logical axioms, inference rules, and formal proof, while we also present some backgrounds in syntax and semantics of equational logic. We will show that equational logic of [29] is sound (with respect to the conventional model of evaluation of Boolean expressions) and complete. Proofs have been presented in either the Hilbert style or the equational style. We explain both styles and argue that the equational style is superior. The equational style makes it possible to develop and present calculations in a rigorous manner, without complexity and detail overwhelming (in contrast to other proof style) (for the details see [14, 15] and also [30]).

2.1 Syntax of Equational Logic

Equational Logic is a formal language, which has a set of symbols (*alphabet*), a set of formation rules (*syntax*) that tells us whether a formula in propositional logic is well-formed formula (grammatically correct), and a *semantics* that assigns formulas a truth value (meaning). It is a natural language, like English. This formal language has been constructed to formulate, for example, the axioms, theorems, and proofs. In that context, the connectives played an important role. Therefore we include the following symbols in the propositional logic languages: "$\neg$" (for “negation”), " $\wedge$ " (for “conjunction”), " $\vee$ " (for “disjunction”), " $\rightarrow$ " (for “implication”), and " $\equiv$ " (as a symbol for “equivalence”), and Boolean constants, namely $\top$ and $\bot$.

Definition 2.1. ([29]) **(Alphabet of Equational Logic language)**

The language of equational logic consists of propositional variables p, q,…, binary connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\equiv$, and Boolean constants, namely $\top$ and $\bot$.

$\mathcal{R}$ shall stand for the language of equational logic.

Definition 2.2. ([29]) **(Equational Logic Formulas)**

All Boolean variables are atomic formulae, and so are the symbols $\top$ and $\bot$. If P and Q are formulae, then so are the following $\neg P$, $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$, and $P \equiv Q$.

Let us we denote by $\mathcal{F}_{\mathcal{R}}$ the set of all formulas for the given language $\mathcal{R}$, and by Γ the special axioms (sometimes also non-logical axioms), that is any subset $\Gamma \subseteq \mathcal{F}_{\mathcal{R}}$.

2.2 Semantics of Equational Logic

The semantics of Boolean formulae is defined through a process that allows us to assign a logical meaning to formulas, and this under certain conditions.

Definition 2.3. ([29]) **(Truth Evaluation)**

A *truth evaluation* e is a function $e : \mathcal{F}_{\mathcal{R}} \rightarrow S,\ S = \{\mathrm{T}, \mathrm{F}\}$ is defined as follows: if $p \in \mathcal{F}_{\mathcal{R}}$ is a propositional variable, then $e(p) \in S$, while $e(\top) =$ T and $e(\bot) =$ F. Furthermore

$$e(\neg P) = \neg e(P)$$

$$e(P \odot Q) = e(P) \odot e(Q), \text{ where } \odot \in \{\wedge, \vee, \rightarrow, \equiv\}.$$

Table 2-1 Truth Table

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$(p \equiv q)$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Definition 2.4. ([29]) **(Truth Tables)**

A *truth-table* is a table for visually displaying the distribution of truth and falsity through a composite formula given the basic inputs from the atomic formulae. There are five functions or operations (Boolean functions), that take values from the set {F, T} as inputs and produce values in the same set as outputs, and Table 2-1 describes their behavior, which known as a *truth table*.

Definition 2.5. ([29]) **(Tautology)**

A formula $P \in \mathcal{F}_{\mathcal{R}}$ is a *tautology* if $e(P) = \mathrm{T}$ for each truth evaluation $e : \mathcal{F}_{\mathcal{R}} \to S$. We use $\vDash_{\text{taut}} P$ as the notation to indicate that P is a tautology.

Example 2.1. ([29]) **(Some tautologies)**

$\top$ and $q \to q$ are tautologies. The latter follows from $e(q \to q) = e(q) \to e(q)$ and Table 2-2.

Table 2-2: Truth table of $(q \to q)$

$\boldsymbol{q}$	$\boldsymbol{q}$	$(\boldsymbol{q} \to \boldsymbol{q})$
T	T	T
F	F	T

2.3 Proofs and Theorems

Equational logic is developed to write down theorems. It is a tool through which we formulate and establish mathematical truth. This truth is captured absolutely (tautologies) or relatively to certain hypotheses (tautological implications). Thus, our main task when we use Boolean logic, is to discover and verify tautologies, and more generally, to discover and verify tautological implications. The process of certifying tautologies and tautological implications is syntactic instead of semantic (truth table driven) and is called *theorem proving.*

First off, axioms are usually statements that are taken to be true. There are two types of axioms: The logical axioms are certain well-chosen absolute truths; therefore, they are tautologies. The other type is called special axioms, also named non-logical axioms or assumptions or hypotheses.

2.3.1 Logical Axioms

Logical axioms codify the most basic properties of the connectives, and describe its behavior. The following list presents the logical axioms for propositional logic (see [29]).

Definition 2.6. (Logical Axioms)

In what follows, P, Q, R denote arbitrary formulae:

(1)	Associativity of $\equiv$	$((P \equiv Q) \equiv R) \equiv (P \equiv (Q \equiv R))$
(2)	Symmetry of $\equiv$	$(P \equiv Q) \equiv (Q \equiv P)$
(3)	$\top$ vs. $\bot$	$\top \equiv \bot \equiv \bot$
(4)	Introduction of $\neg$	$\neg P \equiv P \equiv \bot$
(5)	Associativity of $\vee$	$(P \vee Q) \vee R \equiv (P \vee (Q \vee R)$
(6)	Symmetry of $\vee$	$P \vee Q \equiv Q \vee P$
(7)	Idempotency of $\vee$	$P \vee P \equiv P$
(8)	Distributivity of $\vee$ Over $\equiv$	$P \vee (Q \equiv R) \equiv P \vee Q \equiv P \vee R$
(9)	Excluded Middle	$P \vee \neg P$
(10)	Golden Rule	$P \wedge Q \equiv P \equiv Q \equiv P \vee Q$
(11)	Implication	$P \rightarrow Q \equiv P \vee Q \equiv Q$

2.3.2 Inference Rules

Inference rule is a logical construct which takes premises, analyzes their syntax and returns a conclusion (deriving new formulas from old ones).

The following two are our Inference Rules of Boolean logic, given with the help of the syntactic variables P, Q, C and **p**[1]**:**

The Leibniz rule (Leib)

---

[1] The symbol **p** is a metavariable for any propositional variable $p, q, \ldots$.

$$\frac{P \equiv Q}{C[\mathbf{p} \coloneqq P] \ \equiv \ C[\mathbf{p} \coloneqq Q]}$$

The Equanimity rule (EA)

$$\frac{P, \ \ P \equiv \ Q}{Q}$$

An instance of an inference rule is obtained by replacing all the letters P, Q, C by specific formulae and **p** by a specific variable.

We call the "numerator" the premises (we also say hypotheses or assumptions) and the "denominator" the conclusion of the rule.

The Leibniz rule (Leib) allows us to "substitute equals for equals" in an expression without changing the value of that expression. It therefore gives a method for demonstrating the equality of two expressions. In this method, the format we use to show an application of Leibniz is

$$\begin{aligned} & C[\mathbf{p} \coloneqq P] \\ \equiv \ & \langle P \equiv Q \rangle \\ & C[\mathbf{p} \coloneqq Q] \end{aligned}$$

The first and third lines are the equal expressions of the conclusion in the Leibniz rule; the annotation on the middle line is the premise "$P \equiv Q$".

Once we have written "$P \equiv Q$", we can choose any formula C whatsoever and any variable **p** and construct the output, first effecting two substitutions and then connecting the results with the connective " $\equiv$ " in the indicated order. Note that the Leibniz rule is not functional: Infinitely many different outputs are possible for a given input "$P \equiv Q$".

Definition 2.7. ([29]) **(Proofs)**

A *proof* is any finite (ordered) sequence of formulae (theorems), where each formula is a *premise* or *logical axiom* or a *derived formula* from earlier sentences in the proof by one of the rules of inference.

The last formula is the theorem (also called goal) that we want to prove.

2.3.3 Equational versus Hilbert-style proofs

A Hilbert-style proof consists of a sequence of formulae written vertically on the page, numbering every row for referring to previous formulae, and provided by annotations to explain what we are doing at every step and why. Each formula is hypnosis or an axiom or the conclusion of an inference rule whose premises appear previously (axioms, or proved theorems). Such formula is called a theorem.

As an example, we give a simple annotated Hilbert proof from [29]:

Example 2.1. ([29]) (the other equanimity)

$$Q,\ P \equiv Q \vdash P.$$

Proof. ([29])

(1)	Q	(hypothesis)
(2)	$P \equiv Q$	(hypothesis)
(3)	$P \equiv Q \equiv Q \equiv P$	(Symmetry of $\equiv$)
(4)	$Q \equiv P$	((2) + (3) + (EA))
(5)	P	((1) + (4) + (EA))

Example 2.2. ([29]) (Transitivity of " $\equiv$ ")

$$P \equiv Q,\ Q \equiv R \vdash P \equiv R.$$

Proof. ([29])

(1) $P \equiv Q$ (hypothesis)

(2) $Q \equiv R$ (hypothesis)

(3) $(P \equiv Q) \equiv (P \equiv R)$ ((2) + Leib; $C-$ part is "$P \equiv \mathbf{p}$", $\mathbf{p}$ is $\mathbf{fresh}$[2])

(4) $P \equiv R$ ((1) + (3) + (EA))

On the other hand, the equational style proof consists of a sequence of formulas of the form $P_1 \equiv P_2,\ P_2 \equiv P_3, \ldots, P_{n-1} \equiv P_n$. Each of the formulas $P_{i-1} \equiv P_i$ must be either an assumption, or a logical axiom, or derived earlier, or derived using the Leibniz inference rule. It consisting of a series of applications of the Leibniz rule is linked implicitly by the transitivity. Each step of the proof is provided by an informative annotation to explain how we arrived at the formula $P_{i-1} \equiv P_i$. The following is the equational style proof layout:

$$\begin{array}{l} P_1 \\ \Leftrightarrow \langle \text{Annotation} \rangle \\ P_2 \\ \vdots \\ P_{n-1} \\ \Leftrightarrow \langle \text{Annotation} \rangle \\ P_n \end{array} \qquad \text{(Equational Style Proof Layout)}$$

Since the symbol " $\equiv$ " is associative, it is not conjunctional; that is "$P \equiv Q \equiv R$" does not mean "$P \equiv Q$" and "$Q \equiv R$"; therefore, the symbol " $\Leftrightarrow$ " is our conjunctional " $\equiv$ " and will appear only in equational proofs and only on their

[2] "Fresh" means that **p** does not occur in any of P, Q, R.

leftmost column at that. Thus "$P \Leftrightarrow Q \Leftrightarrow R$" means only "$P \equiv Q$" and "$Q \equiv R$". It is meant that $P_1 \equiv P_2$ and $P_2 \equiv P_3$ and $P_3 \equiv P_4$, etc.

When using Leibniz we must be also very clear as to what the "C-part" is and state any special requirements that we may have put on **p**, e.g., "freshness". For Leibniz, the suggested style of annotation is

$$\text{Leib} + \begin{Bmatrix} \text{Axiom} \\ \text{Hypothesis} \\ \text{Theorem} \end{Bmatrix}; \text{"}C - \text{part" ...}$$

We now present the equational style proof for Example 2.1.

$$\begin{aligned} & P \\ \Leftrightarrow & \langle \text{hypothesis } (P \equiv Q) \rangle \\ & Q \end{aligned}$$

Example 2.3. ([29]) $\vdash P \equiv P$.

Proof.

$$\begin{aligned} & P \vee P \equiv P \\ \Leftrightarrow & \langle (\text{Leib}) + \text{Axiom}: P \vee P \equiv P; \text{"C} - \text{part"}: \mathbf{p} \equiv P \rangle \\ & P \equiv P \end{aligned}$$

Example 2.4. ([29]) $\vdash P \vee \top$.

Proof. ([29])

$$\begin{aligned} & P \vee \top \\ \Leftrightarrow & \langle (\text{Leib}) + \text{Axiom}: \top \equiv \bot \equiv \bot; \text{"C} - \text{part"}: P \vee \mathbf{p} \rangle \\ & P \vee (\bot \equiv \bot) \\ \Leftrightarrow & \langle \text{Axiom } (P \vee (Q \equiv R) \equiv P \vee Q \equiv P \vee R) \rangle \\ & P \vee \bot \equiv P \vee \bot \end{aligned}$$

Remark 2.1.

(1) The first formula of equational style proof is equivalent to the last one. Thus, the equational proof need not be built up to the final formula as in the case of Hilbert-style proof; whenever convenient, it can start with it and end up with some known formula as in Example 2.4. Moreover, each step is an application of Leibniz and we need not to mention none of the inference rules explicitly in an equational proof, this reduces the amount of writing when presenting the proof and the amount of reading in understanding it. Consequently, the proofs are more concise and thus, easy to read and remember (for more details see [29] or [14]).

(2) In the equational style proof, the aim of each step is to replace the expression using Leibniz (substitution of equals by equals). The shape of the expression and the already existing theorems give guidance to construct the proofs easily and then to remember it. Furthermore, making it possible to teach its development.

Many theorems, which describe the main properties of the propositional logic, have proofs were introduced in [29].

2.4 Soundness and completeness of propositional logic

Syntax and semantics are two parts of propositional logic. Soundness and completeness theorems for propositional logic show the interplay between these two components. The first states that our logic is truthful, or sound. That is, whenever $\Gamma \vdash P$, then also $\Gamma \vDash_{\text{taut}} P$ (i.e. each provable formula is a Boolean tautology). The second states that the chosen axioms (and inference rules) are "just the right ones" to ensure that syntactic proofs are able to generate all tautologies. That is, whenever $\Gamma \vDash_{\text{taut}} P$, then also $\Gamma \vdash P$ (i.e. each true formula is provable).

2.4.1 Soundness

Propositional logic is sound with respect the standard interpretation. To see this, first, prove that if premises of each inference rule are valid then so is its conclusion. Second, check that each axiom is valid, and this is justified by truth tables.

Lemma 2.1. ([29])

The two inference rules preserve truth. That is,

$$P, P \equiv Q \vDash_{taut} Q$$

, and

$$P \equiv Q \vDash_{\mathrm{taut}} R[\mathbf{p} := P] \equiv R[\mathbf{p} := Q]$$

Theorem 2.1. ([29]) **(Soundness of Propositional Calculus)**

$$\Gamma \vdash P \text{ implies that } \Gamma \vDash_{\mathrm{taut}} P.$$

2.4.2 Completeness

It is shown that propositional logic is complete in [29]. Completeness means that every semantically valid formula can be proved syntactically. There are two methods of proofs. The first one is straightforward. It shows how one can use the hypothesis that a formula P is a tautology in order to construct its formal proof. The second proof shows how one can deduce that a formula P is not a tautology from the fact that it doesn't have a proof. It is hence called a contrapositive construction method. The term contrapositive refers to an implication. The contrapositive of the formal implication "$P \rightarrow Q$" is "$\neg Q \rightarrow \neg P$", therefore proving "$\vdash P \rightarrow Q$" is as good as proving $\vdash \neg Q \rightarrow \neg P$ by Equanimity. The last methodology is used in [29] to prove the completeness of propositional logic.

The proof idea of completeness of propositional Logic in [29] is based on a few constructions along with a few claims and their proofs as follows:

First of all, assume the hypothesis side, $\Gamma \nvdash P$. Then construct a set of formulae, Λ which is as large as possible with the properties that it includes Γ, but also $\Lambda \nvdash P$. Λ is so big a set of assumptions that anything you can prove from them, with any proof, can also be proved by a proof of length one.

We also, define a state v by setting, for each variable $\mathbf{p}, v(\mathbf{p}) = \mathrm{T}$ iff $\mathbf{p} \in \Lambda$; which represents our Main Claim:

For all formulae $P, v(P) = \mathrm{T}$ iff $P \in \Lambda$ (equivalently,$v(P) = \mathrm{F}$ iff $P \notin \Lambda$)

Then, our goal is to prove this claim. The proof is by induction on the complexity of P.

After that, we can easily conclude the proof as follows: by the Main Claim, every formula P in Λ and hence every formula P in Γ since $\Gamma \subseteq \Lambda$ satisfies $v(P) = \mathrm{T}$. On the other hand, as $\Lambda \nvdash P$ it must be $P \notin \Lambda$; thus, again via the Main Claim, $v(P) = \mathrm{F}$. Therefore $\Gamma \nvDash P$. This completes the proof.

Theorem 2.2. ([29]) **(Completeness of Propositional Calculus)**

$$\Gamma \vDash_{\text{taut}} P \text{ implies that } \Gamma \vdash P.$$

Theorem 2.3. ([29]) **(Deduction Theorem)**

For each theory T, formula P and arbitrary formula Q it holds that:

$$T \cup \{P\} \vdash Q \text{ iff } T \vdash P \Rightarrow Q.$$

Chapter 3
EQ-Logics: Fuzzy Logics Based on Fuzzy Equality

When tracing back the development of logic we can distinguish two basic directions: (a) implication is the basic connective and modus ponens is the fundamental inference rule and (b) logical equivalence (taken as an equality between truth values) is the basic connective and the basic inference rules are equanimity and Leibniz ones. Direction (a) is popular than (b) for many years; but the latter, however, gains gradually still more and more interest, too (cf., e.g. [14, 29]). There are at least two main reasons for that. First, equality (equivalence) seems to be more essential connective than implication. This direction is justified by the idea presented by G.W.Leibniz (cf. [2]) that a fully satisfactory logical calculus must be an equational one. Moreover, the formal proofs can be more effectively formed in an equational style. The second reason is also argued by its proponents (see, for example, [14]) that equational logic is the pedagogically proper setting to do proofs because its main tool, substitution of equals for equals, makes it easier to discover proofs (than it is when using the Hilbert style of deduction), rendering proofs more natural and more calculational.

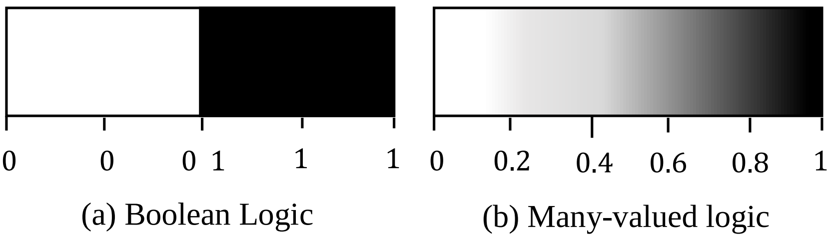

Figure 3-1 Boolean logic versus Many-valued logic

The restriction of classical logic is that every proposition either completely true or completely false (no middle). However, there are also propositions with variable answers. The following example shows how a classical argument fails to work when one passes from classical logic to Multi-valued logic: The sentence "The patient is young" is true to some degree. The lower the age of the patient (measured e.g. in years), the more the sentence is true. Figure 3.1 shows that the truth of a many-valued proposition is a matter of degree.

Classical logic is just a special case of many-valued logic and of course fuzzy logic, so the "fuzziness" can be restricted. In other words, when many-valued logic is restricted to the values zero and one (true, and false), it becomes classical logic. So, if we restrict each connective in many-valued logic to zero and one, it becomes classical connective.

As in the classical logic, there are two basic directions in many-valued logics. First, logics based on implication while (fuzzy) equality is derived from it. EQ-logics whose EQ-algebras as the algebraic semantics is an example of this direction. The second direction is based on (fuzzy) equality instead of implication, for example the basic logic (BL) which has semantical domain of the residuated lattice (see [17]). These directions generalizes the corresponding directions in classical logic. Unlike classical logic, which can be equivalently developed starting either by implication or by equivalence (cf.[29]), many-valued logics, however, the situation is different; implication based and equality based approaches are no more equivalent; i.e. the fuzzy EQ-logic is not equivalent with the residuated fuzzy logics.

In this chapter, we present a specific developed formal logic in which the fuzzy equality is basic connective and the implication is derived from it. Moreover, the fusion connective (strong conjunction) is non-commutative. This logic is called EQ-logic and can be considered as special type of fuzzy logic (cf. [6]) and a generalization of the equational classical logics due to Gries and Schneider [14]. First, we introduce of the concept of EQ-algebra and its main properties as well as the corresponding propositional EQ-logics and show its main properties including the completeness property. Furthermore, we show the important effect of adding **Δ**-connective to EQ-logic language and how it is necessary to develop its first-order version (cf. [5]). Finally, we present the concept and properties of prelinear $EQ_{\boldsymbol{\Delta}}$-Logic.

3.1 EQ-Algebras: The Algebraic Semantics of EQ-Logics

Each many-valued logic is uniquely defined by the algebraic properties of its truth values structure. It is generally for many years accepted that this algebraic structure must be a residuated lattice in fuzzy logic, possibly fulfilling some extra properties (the definition and several useful properties of residuated lattices can be found in [12]). Unlike the stated direction in algebraic semantics where multiplication and residuation are the basic operations, and the most important connectives are strong conjunction and implication in the corresponding fuzzy logics, there is a new direction in the development of logic justified by G.W. Leibniz's idea (cf. [2]). Hence, as an alternative to residuated lattices, a special algebra called EQ-algebra has been presented by Novák [22] and elaborated in [23]. The original motive was to present a special algebra of truth values for fuzzy type theory (FTT) (see [21]) that generalizes the classical type theory (cf. [1]) where the basic connective is equality instead of implication. Analogously, the main connective in FTT should be fuzzy equality " $\sim$ ". Another motive for EQ-algebras arises from the equational style of proof in logic.

From the point of view of logic, the basic difference between residuated lattices and EQ-algebras lies in how the implication operation is obtained. Where in residuated lattices, it is obtained from a (strong) conjunction, in EQ-algebras, it is derived from fuzzy equality. As well as, EQ-algebras behave differently than residuated lattices, as is shown (see [8]) by the fact that $p \rightarrow q = \mathbf{1}$ doesn't imply that $p \leq q$. Therefore, the two kinds of algebras differ in multiple basic points, although their many similar or matching properties. Indeed, EQ-algebras generalize residuated lattices since they relax the tie between multiplication and residuation, the so-called adjointness property (i.e. between conjunction and implication in logic); the implication is defined from the fuzzy equality " $\sim$ " by the formula $p \to q = (p \wedge q) \sim p$. Since this equation holds also for the biresiduum, every residuated lattice can be considered as an EQ-algebra but not vice versa, see Example 3.2.

3.1.1 Residuated lattices

Definition 3.1. ([23])

An algebra $\mathcal{L} = (L,\ \wedge,\ \vee,\ \otimes,\ \Rightarrow, \mathbf{0}, \mathbf{1})$ of type (2, 2, 2, 2, 0, 0) is called a commutative, integral, bounded *residuated lattice* if the following conditions are satisfied:

(L1) $(L,\ \wedge,\ \vee, \mathbf{0}, \mathbf{1})$ is a lattice with the bottom and top elements $\mathbf{0}$ and $\mathbf{1}$, respectively (with respect to the lattice ordering " $\leq$ "),

(L2) $(L,\ \otimes, \mathbf{1})$ is a commutative monoid with the unit element $\mathbf{1}$,

(L3) $\otimes$ and $\Rightarrow$ form an adjoint pair, i.e. for all $p, q, r \in L$ it holds that

$$p \otimes q \leq r \text{ iff } p \leq q \Rightarrow r \qquad \text{(Adjointness property)}$$

The binary operation " $\wedge$ " is called meet, " $\vee$ " is called join, " $\otimes$ " is called multiplication, and " $\Rightarrow$ " is called residuation.

There are many properties of residuated lattices, see [12, 24]. In the following definition, we shall introduce an algebra called BL-algebra (a residuated lattice, fulfilling some additional properties) which is the algebraic semantics of the basic many-valued logic, BL; that is considered, actually an example of fuzzy logic based on the implication as a basic connective instead equivalence (for, more details see, [17]).

Definition 3.2. ([23])

A residuated lattice $\mathcal{L} = (L, \wedge, \vee, \otimes, \Rightarrow, \mathbf{0}, \mathbf{1})$ is a *BL-algebra* iff the following two identities hold for all $p, q \in L$:

(a) *Prelinearity*: $(p \Rightarrow q) \vee (q \Rightarrow p) = \mathbf{1}$;
(b) *Divisibility*: $p \otimes (p \Rightarrow q) = p \wedge q$.

MTL-algebras are residuated lattices fulfilling the prelinearity condition. They are the algebraic semantics of the Monoidal t-norm based logic (or shortly, MTL) (for, more details see, [10]).

3.1.2 EQ-algebras

I. Definition and Fundamental Properties of EQ-algebras

Definition 3.3. ([8])

An algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, \mathbf{1})$ of type (2, 2, 2,0) is called an EQ-algebra where for all $p, q, r, s \in E$:

(E1) $(E, \wedge, \mathbf{1})$ is a $\wedge$-semilattice with top element $\mathbf{1}$. We set $p \leq q$ iff $p \wedge q = p$;
(E2) $(E, \otimes, \mathbf{1})$ is a monoid and $\otimes$ is isotone in both arguments w.r.t. $p \leq q$,
(E3) $p \sim p = \mathbf{1}$; (reflexivity)
(E4) $((p \wedge q) \sim r) \otimes (s \sim p) \leq r \sim (s \wedge q)$; (substitution)
(E5) $(p \sim q) \otimes (r \sim s) \leq (p \sim r) \sim (q \sim s)$; (congruence)
(E6) $(p \wedge q \wedge r) \sim p \leq (p \wedge q) \sim p$; (monotonicity)
(E7) $p \otimes q \leq p \sim q$.

The binary operation "$\wedge$" is called meet (infimum), "$\otimes$" is called multiplication, and "$\sim$" is a fuzzy equality.

The substitution axiom (E4) is motivated by the substitution principle formulated already by G.W. Leibniz: “if P equals Q then P can be replaced by Q wherever P occurs”. The congruence axiom naturally generalizes the following property of the classical equality: if $p = q$ and $r = s$, then the truth of $p = r$ is the same as the truth of $q = s$.

Remark 3.1. ([8])

The definition of EQ-algebras in [[23], Definition 1] includes extra axiom, namely,

$$(p \wedge q) \sim p \leq (p \wedge q \wedge r) \sim (p \wedge r) \tag{3.1}$$

It has been shown in [8] that we do not need this axiom because it is derived from the other axioms. Moreover, Definition 3.3 differs from the original definition of EQ-algebras ([23], Definition 1) in that the multiplication "$\otimes$" need not be commutative. Also, that the commutativity axiom of multiplications is superfluously restrictive, i.e. a weaker requirement put on non-commutative multiplications is sufficient to guarantee all expected general properties of fuzzy equalities and EQ-algebras.

Clearly, " $\leq$ " is the classical partial order. We set, for $p, q \in E$:

$$p \rightarrow q = (p \wedge q) \sim p \tag{3.2}$$

$$\tilde{p} = p \sim \mathbf{1} \tag{3.3}$$

If $\mathcal{E}$ also contains a bottom element $\mathbf{0}$, then we define the unary operation $\neg$ on E by

$$\neg p = p \sim \mathbf{0} \qquad p \in E \tag{3.4}$$

The derived operation (3.2) is called *implication*. Hence, we may rewrite (E6) and (3.1) as

$$p \to (q \wedge r) \leq p \to q \tag{3.5}$$

$$p \to q \leq (p \wedge r) \to q \tag{3.6}$$

We will introduce the essential properties of EQ-algebras presented in ([8, 19, 23]).

Lemma 3.1. ([8, 19, 23])

Let $\mathcal{E}$ be an EQ-algebra. For all $p, q, r \in E$, it holds that:

(a) $p \sim q = q \sim p$; (symmetry)

(b) $(p \sim q) \otimes (q \sim r) \leq (p \sim r)$; (transitivity)

(c) $(p \sim s) \otimes ((p \wedge q) \sim r) \leq (s \wedge q) \sim r$;

(d) $(p \wedge q) \sim p \leq (p \wedge q \wedge r) \sim (p \wedge r)$;

(e) Let $p \leq q$, then

$p \to q = \mathbf{1}, p \sim q = q \to p, r \to p \leq r \to q$ and $q \to r \leq p \to r$;

(f) $(p \to q) \otimes (q \to r) \leq (p \to r)$; (transitivity of implication)

(g) $p \otimes q \leq p \wedge q \leq p, q$ and $q \otimes p \leq p \wedge q \leq p, q$;

(h) $(p \sim q) \leq p \to q$ and $p \to p = \mathbf{1}$; ($\to$ is reflexive)

(i) $p = q$ implies $p \sim q = \mathbf{1}$;

(j) $q \leq \tilde{q} \leq p \to q$;

(k) $p \overset{0}{\leftrightarrow} q \leq (p \sim q) \leq p \leftrightarrow q$; If $\mathcal{E}$ is linearly ordered, then $\leq$ can be replaced by an equality;

(l) $p \to s \leq (r \to p) \to (r \to s)$;

(m) $p \to s \leq (s \to r) \to (p \to r)$;

(n) $p \to q = p \to (p \wedge q)$;

(o) $p \to (q \to r) \leq q \to (p \to \tilde{r})$;

(p) $p \to (q \to r) \leq (p \otimes q) \to \tilde{r}^4$.

From here on, we shall often freely use the transitivity and symmetry of "$\sim$" without special reference to the above lemma.

Let us put

$$p \leftrightarrow q = (p \to q) \wedge (q \to p). \tag{3.7}$$

$$p \overset{0}{\leftrightarrow} q = (p \to q) \otimes (q \to p) \tag{3.8}$$

Theorem 3.1. ([8])

The class of EQ-algebras is a variety.

Definition 3.4. ([23])

Let $\mathcal{E}$ be an EQ-algebra. We say that it is:

- *Separated* if for all $p, q \in E$,

$$p \sim q = \mathbf{1} \text{ implies } p = q \tag{3.9}$$

- *Spanned* if it contains a bottom element $\mathbf{0}$ and

$$\tilde{\mathbf{0}} = \mathbf{0} \sim \mathbf{1} = \mathbf{0} \tag{3.10}$$

- *Good* if for all $p \in E$,

$$\tilde{p} = p \tag{3.11}$$

- *Residuated* if for all $p, q, r \in E$,

$$(p \otimes q) \wedge r = (p \otimes q) \text{ iff } p \wedge ((q \wedge r) \sim q) = p \tag{3.12}$$

- *Lattice-ordered* EQ-algebra if the underlying $\wedge$-semilattice is a lattice,
- *Lattice* EQ-algebra (ℓEQ-algebra) if it is lattice-ordered in which the following substitution axiom holds for all $p, q, r, s \in E$:

$$((p \vee q) \sim r) \otimes (s \sim p) \leq r \sim (s \vee q) \tag{3.13}$$

- *prelinear* if for all $p, q \in E$, 1 is the unique upper bound in E of the set

$$\{(p \to q), (q \to p)\}.$$

Remark 3.2. ([23])

(i) Every good EQ-algebra is obviously spanned but not vice versa.

(ii) Clearly, (3.12) can be written in a classical way such as

$$p \otimes q \leq r \text{ iff } p \leq q \to r.$$

(iii) An EQ-algebra can be lattice-ordered but not necessarily an ℓEQ-algebra.

(iv) The prelinearity does not require the existence of a join operator in E. However, in the following, we will illustrate that every prelinear and good EQ-algebra is a lattice-ordered one where the join operation is definable in terms of the meet " $\wedge$ " and the implication " $\to$ " operations.

II. Examples of EQ-algebras

In this section, we introduce a few interesting examples of EQ-algebras.

Example 3.1. ([23])

Let $\mathcal{L} = (L, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1})$ be a residuated lattice.

(a) The algebra $\mathcal{L}' = (L, \wedge, \otimes, \leftrightarrow, \mathbf{1})$ is a separated EQ-algebra. If $\mathcal{L}$ is linearly ordered (then $\leftrightarrow = \overset{0}{\leftrightarrow}$ according to Lemma 3.1(k)), then also $\mathcal{L}'' = (L, \wedge, \otimes, \overset{0}{\leftrightarrow}, \mathbf{1})$ is a separated EQ-algebra.

(b) Let $\odot \leq \otimes$ be an isotone monoidal operation on L. Then also $\mathcal{L}' = (L, \wedge, \odot, \leftrightarrow, \mathbf{1})$ is a separated EQ-algebra.

Example 3.2. ([7])

Example of a finite non-trivial good EQ-algebra is the following: its (semi)lattice structure is in Figure 3.2. Fuzzy equality and multiplication are defined as in Table 3.1 and Table 3.2 respectively.

Table 3-1 Fuzzy equality of Example 3.2

$\sim$	**0**	p	q	r	s	t	u	**1**
0	**1**	t	u	s	r	p	q	**0**
p	t	**1**	s	u	r	p	r	p
q	u	q	**1**	t	r	r	q	q
r	s	u	t	**1**	r	r	r	r
s	r	r	r	r	**1**	u	t	s
t	p	p	r	r	u	**1**	s	t
u	q	c	q	r	t	s	**1**	u
1	**0**	p	q	r	s	t	u	**1**

Table 3-2 Multiplication of Example 3.2

$\otimes$	**0**	p	q	r	s	t	u	**1**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
p	**0**	**0**	**0**	**0**	**0**	**0**	**0**	p
q	**0**	**0**	**0**	**0**	**0**	**0**	**0**	q
r	**0**	**0**	**0**	**0**	**0**	**0**	p	r
s	**0**	**0**	**0**	**0**	s	s	s	s
t	**0**	**0**	**0**	**0**	s	e	s	t
u	**0**	**0**	**0**	**0**	s	d	s	u
1	**0**	p	q	r	s	e	u	**1**

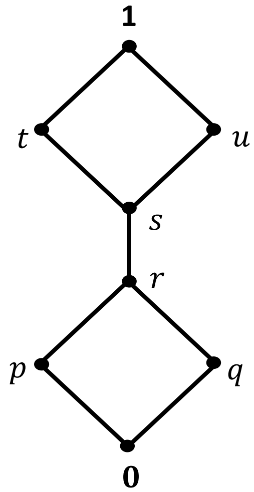

Figure 3-2 Eight elements good EQ-algebra

Since $r \otimes f = p$ but $f \otimes r = \mathbf{0}$, the multiplication is not commutative. Moreover, this algebra is non-residuated since, e.g., $\mathbf{0} = p \otimes u \leq q$; but $p \nleq u \to q = q$.

III. Properties of special EQ-algebras

Proposition 3.1. ([8])

The following statements are equivalent:

(a) An EQ-algebra $\mathcal{E}$ is *separated*.

(b) $p \leq q$ iff $p \to q = \mathbf{1}$ for all $p, q \in E$.

Remark 3.3. ([8])

According to the last proposition the implication operation " $\to$ " in a separated EQ-algebra precisely reflects the ordering " $\leq$ ".

Proposition 3.2. ([8])

Let $\mathcal{E}$ be a lattice-ordered EQ-algebra, then the following hold $\forall\, p, q, r \in E$:

(a) $\mathcal{E}$ is ℓEQ-algebra if and only if the following inequality holds,

$$p \sim q \leq (p \vee r) \sim (q \vee r) \qquad (3.14)$$

(b) $p \wedge q \to r = (p \to r) \vee (q \to r)$.

Proposition 3.3. ([8, 23])

Let $\mathcal{E}$ be an ℓEQ-algebra, then the following hold for all $p, q, r \in E$:

(a) $p \to q = (p \vee q) \sim q = (p \vee q) \to q$;

(b) $(p \to r) \otimes (q \to r) \leq (p \vee q) \to r$.

Lemma 3.2. ([7])

Let $\mathcal{E}$ be a prelinear and separated EQ-algebra. Then for all $p, q, r, s \in E$, it holds that:

(a) $p \leftrightarrow q = p \sim q$;

(b) $p \to (q \wedge r) = (p \to q) \wedge (p \to r)$.

Lemma 3.3. ([7])

Let $\mathcal{E}$ be a prelinear and separated ℓEQ-algebra; then the following hold for all $p, q, r \in E$:

(a) $\mathcal{E}$ is distributive; i.e.,

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

(b) $(p \vee q) \to r = (p \to r) \wedge (q \to r)$.

Note that the dual of the identity in Lemma 3.3(a) (i.e., $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ holds and the two identities are equivalent to each other (see [28]).

Proposition 3.4. ([8]) The following statements are equivalent:

(a) An EQ-algebra E is *good*.

(b) $\mathbf{1} \to q = q$ for all $q \in E$.

Lemma 3.4. ([8, 23])

Let $\mathcal{E}$ be a good EQ-algebra. For all $p, q, r \in E$, it holds that

(a) $p \leq (p \sim q) \sim q$;

(b) $\mathcal{E}$ is separated and axiom (E7) is provable from the other EQ-axioms;

(c) $p \leq (p \to q) \to q$;

(d) $p \otimes (p \sim q) \leq p \wedge q$ and $(p \sim q) \otimes p \leq p \wedge q$;

(e) $p \otimes (p \to q) \leq p \wedge q$ and $(p \to q) \otimes p \leq p \wedge q$;

(f) $p \leq q \to r$ implies $p \otimes q \leq r$ and $q \otimes p \leq r$.

The following theorem presented in [8] and which shows that $\{\to, \mathbf{1}\}$-reducts[3] of good EQ-algebras are BCK-algebras (for the definitions and fundamental properties of BCK-algebras, (see [13, 18, 26, 27]). Thus, each good EQ-algebra can be regarded as a BCK-meet-semilattice with the additional operations "$\otimes$" and " $\sim$ ".

Theorem 3.2. ([8])

The $\{\wedge, \to, \mathbf{1}\}$-reducts of good EQ-algebras are BCK-meet-semilattices, where " $\to$ " is defined by (3.2).

Consequently, the proof of the following lemma follows from the theory of BCK-algebras well-known results.

Lemma 3.5. ([8, 23])

Let $\mathcal{E}$ be a good EQ-algebra. For all $p, q, r \in E$, it holds that

(a) $p \leq q \to r$ iff $q \leq p \to r$;

(b) $p \to (q \to r) = q \to (p \to r)$; (Exchange principle (EP))

(c) $p \to (q \to r) \leq (p \otimes q) \to r$ and $p \to (q \to r) \leq (q \otimes p) \to r$;

(d) For all indexed families $\{p_i\}$ in E, provided that $\{p_i\}$ has supremum in E, we have

$$\bigvee_i p_i \to r = \bigwedge_i (p_i \to r).$$

[3] Given an algebra (G, H) where H is the set of operations on G, and $H' \subseteq H$: Then the algebra (G, H') is called the H'-reduct of (G, H).

Theorem 3.3. ([7])

Let $\mathcal{E}$ be a prelinear and good EQ-algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, \mathbf{1})$, then $\mathcal{E}$ is a prelinear and good ℓEQ-algebra, where the join operation is given by

$$p \vee q = ((p \to q) \to q) \wedge ((q \to p) \to p) \quad p, q \in E \tag{3.15}$$

Remark 3.4.

As a result of Theorem 3.3, all good ℓEQ-algebras properties are also prelinear and good EQ-algebras properties (for the properties of good ℓEQ-algebras, see [9, 23]).

Proposition 3.5. ([7])

The following holds in prelinear and good EQ-algebra $\mathcal{E}$ for all $p, q \in E$:

(a) $p \vee q = \mathbf{1}$ iff $p \to q = q$ and $q \to p = p$**;**

(b) $p \overset{0}{\leftrightarrow} q = p \sim q$ iff $p \vee q = \mathbf{1}$ implies $p \otimes q = p \wedge q$.

Remark 3.5.

In general, in a prelinear and good (commutative) EQ-algebra $p \overset{0}{\leftrightarrow} q \neq p \sim q$ (see Example 3.3). But, this identity always holds for all linearly ordered EQ-algebras. This shows that prelinearity alone does not characterize the representable class of all good (commutative) EQ-algebras.

Example 3.3. ([7])

Let E be the bounded lattice $\{\mathbf{0}, p, q, r, \mathbf{1}\}$ with the partial order " $\leq$ " defined by:$\mathbf{0} \leq p \leq q \leq \mathbf{1}$ and $\mathbf{0} \leq p \leq r \leq \mathbf{1}$, whereas q and r are non-comparable as shown in Figure 3-3.

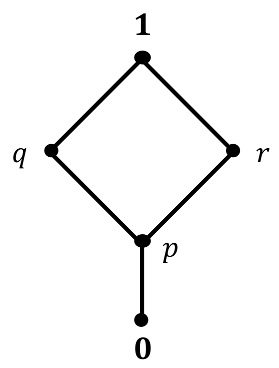

Figure 3-3 Bounded lattice $\{\mathbf{0}, p, q, r, \mathbf{1}\}$

The following fuzzy equality and multiplication define a prelinear and good EQ-algebra in which the identity $p \overset{0}{\leftrightarrow} q = p \sim q$ does not hold for all $p, q \in E$, since, e.g. $p = q \sim r \neq (q \to r) \otimes (r \to q) = r \otimes q = \mathbf{0}$.

Table 3-3 Multiplication of Example 3.3

$\otimes$	$\mathbf{0}$	p	q	r	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
p	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	p	p
q	$\mathbf{0}$	p	q	p	q
r	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	r	r
$\mathbf{1}$	$\mathbf{0}$	p	q	r	$\mathbf{1}$

Table 3-4 Fuzzy equality of Example 3.3

$\sim$	$\mathbf{0}$	p	q	r	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
p	$\mathbf{0}$	$\mathbf{1}$	p	p	p
q	$\mathbf{0}$	p	$\mathbf{1}$	p	q
r	$\mathbf{0}$	p	p	$\mathbf{1}$	r
$\mathbf{1}$	$\mathbf{0}$	p	q	r	$\mathbf{1}$

Lemma 3.6. ([7])

A good EQ-algebra $\mathcal{E}$ is prelinear if and only if the following inequality holds for all $p, q, r \in E$:

$$(p \to q) \to r \leq ((q \to p) \to r) \to r \tag{3.16}$$

Inequality (3.16) has been chosen by Hájek and El-Zekey (see [7, 17]) as the prelinearity axiom in his axiomatization of BL-algebras and good EQ-algebras, respectively, obviously because it is free from operations of lattice.

Definition 3.5. ([8])

Let $\mathcal{E} = (E, \wedge, \otimes, \sim, \mathbf{1})$ be a separated EQ-algebra. A subset $F \subseteq E$ is called a *prefilter* of $\mathcal{E}$ if for all $p, q \in E$:

(a) $\mathbf{1} \in F$;

(b) If $p, p \to q \in F$, then $q \in F$.

A prefilter F is said to be *filter* if for all $p, q, r \in E$:

(c) If $p \to q \in F$, then $(p \otimes r) \to (q \otimes r) \in F$, $(r \otimes p) \to (r \otimes q) \in F$.

A prefilter F is called *proper* if $F \neq E$. If $\mathbf{0} \in E$ then a prefilter $F \subset E$ is proper iff $\mathbf{0} \notin F$.

A prefilter F is said to be a *prime* prefilter (or simply *prime*) if for all $p, q \in E$: $p \to q \in F$ or $q \to p \in F$.

It is easy to see that the singleton $\{\mathbf{1}\}$ is a filter in any separated EQ-algebra, and it is contained in any other filter. Note that if F is prime and G is a prefilter such that $F \subseteq G$; then G is a prime prefilter.

Definition 3.6. ([28])

Let $\boldsymbol{P}$ be an algebra of type $\mathcal{F}$. Then thee relation θ is a congruence on $\boldsymbol{P}$ if θ is equivalence relation and satisfies the following compatibility property:

For each n-ary operation (or function) symbol $f \in \mathcal{F}$ and elements $p_i, q_i \in P$, if $p_i \theta q_i$ holds for $1 \leq i \leq n$ then

$$f^{\boldsymbol{P}}(p_1, p_2, \dots, p_n)\, \theta f^{\boldsymbol{P}}(q_1, q_2, \dots, q_n)$$

IV. Representable good EQ-algebras

Recall that an EQ-algebra that is a subdirect product of those with underlying linear order is said to be representable. We assign this section to introduce the characterization of the representable class of good EQ-algebras. This is mainly based on study in-depth the prefilters, filters and the congruences of EQ-algebras and other useful results, leading to this characterization.

Definition 3.7. ([28])

An algebra P is a subdirect product of an indexed family $\{P_i\}_{i\in I}$ of algebras if

(a) $P \leq \prod_{i\in I} P_i$ (i.e. P is a subalgebra of $\prod_{i\in I} P_i$);

(b) $\pi_j(P) = P_j$ for all $j \in I$, where $\pi_j: \prod_{i\in I} P_i \to P_j$ is a natural projection map.

A one-to-one homomorphism $h: P \to \prod_{i\in I} P_i$ is called a subdirect embedding if $h(P)$ is a subdirect product of the family $\{P_i\}_{i\in I}$.

Remark 3.6.

We know that the underlying poset E of an EQ-algebra $\mathcal{E}$ need not be a join-semilattice. So, given $p, q \in E$, we shall write $p \vee q = \mathbf{1}$ to mean that the supremum of $\{p, q\}$ in E, exists and is equal to $\mathbf{1}$.

Proposition 3.7. ([7])

Let $\mathcal{E}$ be good EQ-algebra. Then the following statements are equivalent, for all $p, q, r, s, u \in E$

(a) $\mathcal{E}$ is prelinear and satisfies the quasi-identity

$$p \vee q = \mathbf{1} \text{ implies } p \vee (s \to (s \otimes (r \to (q \otimes r)))) = \mathbf{1} \quad (3.17)$$

(b) $\mathcal{E}$ satisfies the identity

$$(p \to q) \vee (s \to (s \otimes (r \to ((q \to p) \otimes r)))) = \mathbf{1} \tag{3.18}$$

(c) $\mathcal{E}$ satisfies

$$(p \to q) \to u \leq (((s \to (s \otimes (r \to ((q \to p) \otimes r)))) \to u) \to u) \tag{3.19}$$

(a) $\mathcal{E}$ satisfies

$$(s \to (s \otimes (r \to ((q \to p) \otimes r)))) \to u \leq ((p \to q) \to u) \to u \tag{3.20}$$

We have introduced some of the auxiliary results, so we can present the main goal as mentioned in the introduction:

Theorem 3.4. ([7])

Let $\mathcal{E}$ be a good EQ-algebra. The following statements are equivalent:

(a) $\mathcal{E}$ is representable.

(b) $\mathcal{E}$ satisfies (3.19), or equivalently (3.20).

Remark 3.7.

Although the representable good EQ-algebra $\mathcal{E}$ can be characterized by any of the (quasi-)identities or inequalities in Proposition 3.7; it was chosen to use the inequality (3.19), or equivalently (3.20), to avoid using " ∨ "; because the underlying poset E of $\mathcal{E}$ don't need to be a join-semilattice.

3.1.3 EQ$_\Delta$-algebras

In [8], good EQ-algebras has been enriched with a unary operation "Δ" (the so-called Baaz delta) fulfilling some additional hypotheses, which is heavily used in fuzzy logic literature. Moreover, it is shown that the characterization theorem holds for the enriched algebra along the lines parallel to the characterization of representable good EQ-algebras (see [7]).

In this section, we will introduce the enriched good EQ-algebras with unary operation "Δ" fulfilling some additional hypotheses as in the following definition:

Definition 3.8. ([8])

An EQ_Δ-algebra is an algebra $\boldsymbol{\mathcal{E}} = (E, \wedge, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ that is a good EQ-algebra with a bottom element $\mathbf{0}$ expanded by a unary operation $\Delta: E \to E$ fulfilling the following axioms[4]:

(EΔ1) $\Delta\mathbf{1} = \mathbf{1}$;
(EΔ2) $\Delta a \leq a$;
(EΔ3) $\Delta a \leq \Delta\Delta a$;
(EΔ4) $\Delta(a \sim b) \leq \Delta a \sim \Delta b$;
(EΔ5) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$;
(EΔ6) If $a \vee b$ and $\Delta a \vee \Delta b$ exist, then $\Delta(a \vee b) \leq \Delta a \vee \Delta b$;
(EΔ7) $\Delta a \vee \neg\Delta a = \mathbf{1}$ (i.e., $\mathbf{1}$ is the unique upper bound in E of the set $\{\Delta a, \neg\Delta a\}$).

Example 3.4. ([5])

Consider $E = \{\mathbf{0}, p, q, r, \mathbf{1}\}$ to be a five-element chain. Then $\boldsymbol{\mathcal{E}} = (E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ is a linearly ordered EQ_Δ-algebra with the fuzzy equality and multiplication defined in Table 3-5 and Table 3-6 respectively.

The "Δ" operation is defined by $\Delta(\mathbf{1}) = \mathbf{1}$ and $\Delta(x) = \mathbf{0}$ otherwise in all linearly ordered EQ-algebras. Obviously, this algebra is non-commutative and non-residuated. Indeed, for example, $r \otimes p \leq \mathbf{0}$ but $r \nleq p \to \mathbf{0} = p$.

[4] The Δ-axioms are from [8]

Table 3-5 Fuzzy equality of Example 3.4

$\sim$	**0**	p	q	r	**1**
0	**1**	p	**0**	**0**	**0**
p	p	**1**	p	p	p
q	**0**	p	**1**	q	q
r	**0**	p	q	**1**	r
1	**0**	p	q	r	**1**

Table 3-6 Multiplication of Example 3.4

$\otimes$	**0**	p	q	r	**1**
0	**0**	**0**	**0**	**0**	**0**
p	**0**	**0**	**0**	**0**	p
q	**0**	**0**	**0**	q	q
r	**0**	**0**	**0**	r	r
1	**0**	p	q	r	**1**

Theorem 3.14. ([8])

Let $\mathcal{E}$ be a good EQ$_\Delta$-algebra. $\mathcal{E}$ is representable iff $\mathcal{E}$ satisfies (3.19), or equivalently (3.20).

3.1.4 Prelinear EQ$_\Delta$-algebras

In this section we introduce a subclass of EQ$_\Delta$-algebras, called prelinear EQ$_\Delta$-algebras, i.e. EQ$_\Delta$-algebras satisfying prelinearity and the following two inequalities, for all $p, q, r \in E$:

(EΔ8) $\Delta(p \sim q) \leq (p \otimes r) \sim (q \otimes r)$

(EΔ9) $\Delta(p \sim q) \leq (r \otimes p) \sim (r \otimes q)$

As it has been presented in [5], the two inequalities (EΔ8) and (EΔ9) are necessary to assure good behavior of the multiplication " $\otimes$ " with respect to the classical equality, and they are surely necessary to develop also predicate EQ$_\Delta$-logic. If we omit "Δ" in (EΔ8) and (EΔ9) then the resulting EQ-algebra becomes residuated (see [8]).

Proposition 3.8. ([5])

The following properties are equivalent in each EQ_Δ-algebra $\mathcal{E}$:

(a) $\mathcal{E}$ is prelinear;

(b) $\mathcal{E}$ satisfies the following identity, for all $p, q \in E$

$$\Delta(p \to q) \vee \Delta(q \to p) = \mathbf{1} \tag{3.21}$$

(c) $\mathcal{E}$ satisfies the following inequality, for all $p, q, r \in E$

$$(\Delta(p \to q) \to r) \leq (\Delta(q \to p) \to r) \to r \tag{3.22}$$

Proposition 3.9. ([5])

The following properties are equivalent in each EQ_Δ-algebra $\mathcal{E}$:

(a) $\mathcal{E}$ satisfies the following inequalities, for all $p, q, r \in E$

$$\begin{aligned} \Delta(p \to q) &\leq (p \otimes r) \to (q \otimes r) \\ \Delta(p \to q) &\leq (r \otimes p) \to (r \otimes q) \end{aligned} \tag{3.23}$$

(b) $\mathcal{E}$ satisfies the following inequalities, for all $p, q, r \in E$

$$\Delta q \leq r \to (q \otimes r) \text{ and } \Delta q \leq r \to (r \otimes q) \tag{3.24}$$

(c) $\mathcal{E}$ satisfies the following inequality, for all $p, q, r, s \in E$

$$\Delta q \leq (s \to (s \otimes (r \to (q \otimes r)))) \tag{3.25}$$

Furthermore, if we suppose that $\mathcal{E}$ is prelinear, then any one of the above inequalities (hence all) is equivalent to both (EΔ8) and (EΔ9).

Definition 3.9. ([5])

A prelinear EQ_Δ-algebras is an algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ that is a good non-commutative and bounded EQ-algebra with a bottom element $\mathbf{0}$ and a top element $\mathbf{1}$ expanded by a unary operation $\Delta: E \to E$ fulfilling the following axioms:

(PΔ1) $\Delta\mathbf{1} = \mathbf{1}$;

(PΔ2) $\Delta p \leq \Delta\Delta p$;

(PΔ3) $\Delta(p \to q) \leq \Delta p \to \Delta q$;

(PΔ4) $(\Delta(p \to q) \to r) \leq (\Delta(q \to p) \to r) \to r$;

(PΔ5) $\Delta q \leq (s \to (s \otimes (r \to (q \otimes r))))$;

(PΔ6) $\Delta p \vee \neg\Delta p = \mathbf{1}$ (i.e., $\mathbf{1}$ is the unique upper bound in E of the set $\{\Delta a, \neg\Delta a\}$).

Corollary 3.1. ([5])

Prelinear EQ_Δ-algebras are exactly EQ_Δ-algebra satisfying prelinearity, (EΔ8) and (EΔ9).

Theorem 3.5. ([5]) (Representation theorem).

Each prelinear EQ_Δ-algebra is representable.

3.2 Basic EQ-Logic

In this section, we present a propositional EQ-logic introduced by M. Dyba and V. Novak [6] which is called *basic*. This logic is the simplest logic based on a special algebra of truth values called good EQ-algebra introduced by V. Novak in [23] as the algebraic semantics.

3.2.1 Syntax of Basic EQ-Logic

Definition 3.10. ([6]) **(Basic EQ-Logic language)**

The basic EQ-logic language consists of propositional variables $p, q, \ldots$, binary connectives $\wedge, \&, \equiv$ and a truth (logical) constant $\top$. Implication is a derived connective defined by:

$$P \Rightarrow Q \coloneqq P \equiv (P \wedge Q) \tag{3.26}$$

Let $\mathcal{T}$ be a language of basic EQ-logic and $F_{\mathcal{T}}$ stands for the set of all formulas for the given language is $\mathcal{T}$.

3.2.2 Logical Axioms and Inference Rules

The following formulae are axioms of the Basic EQ-logic which are introduced in [6]:

(A1) $(P \equiv \top) \equiv P$

(A2) $P \wedge Q \equiv Q \wedge P$

(A3) $(P \square Q) \square R \equiv P \square (Q \square R)$ where $\square \in \{\&, \wedge\}$

(A4) $P \wedge P \equiv P$

(A5) $(\top \& P) \equiv P$

(A6) $(P \square \top) \equiv P$ where $\square \in \{\&, \wedge\}$

(A7) $((P \wedge Q) \& R) \Rightarrow (Q \& R)$

(A8) $(R \& (P \wedge Q)) \Rightarrow (R \& Q)$

(A9) $(((P \wedge Q) \equiv R) \& (S \equiv P) \Rightarrow (R \equiv (S \wedge Q)))$

(A10) $(P \equiv Q) \& (R \equiv S) \Rightarrow ((P \equiv R) \equiv (S \equiv Q))$

(A11) $(P \Rightarrow (Q \wedge R)) \Rightarrow (P \Rightarrow Q)$

The Inference rules of Basic EQ-logic are Leibniz rule (Leib) and Equanimity rule (EA).

A *theory* T over Basic EQ-logic is any subset $T \subseteq F_{\mathcal{T}}$ of formulas called *special axioms* (also *non-logical axioms*). $T \vdash A$ denotes the sentence "P is provable in T" or "T proves P".

3.2.3 Fundamental Properties of Basic EQ-logic

The following lemma illustrate the fundamental properties of the basic EQ-logic that have been presented in [6].

Lemma 3.7. ([6])

The following properties hold in the basic EQ-logic:

(a) $P \vdash P \equiv \top$, and $P \equiv \top \vdash P$ (Rule (T1),(T2) respectively)

(b) $P \wedge S \equiv R, P \equiv Q \vdash Q \wedge S \equiv R$ (Rule (C))

(c) $(P \equiv S) \equiv R, P \equiv Q \vdash (Q \equiv S) \equiv R$ (Rule (D))

(d) $P \& S \equiv R, P \equiv Q \vdash Q \& S \equiv R$ (Rule (E))

(e) $S \& P \equiv R, P \equiv Q \vdash S \& Q \equiv R$ (Rule (F))

3.2.4 Semantics of Basic EQ-Logic

Definition 3.11. ([6])

A *truth evaluation* $e: F_{\mathcal{T}} \to E$ is defined as follows: if $p \in F_{\mathcal{T}}$ is a propositional variable, then $e(p) \in E$, Furthermore,

$$e(\top) = \mathbf{1};$$
$$e(P \wedge Q) = e(P) \wedge e(Q);$$
$$e(P \& Q) = e(P) \otimes e(Q);$$
$$e(P \equiv Q) = e(P) \sim e(Q).$$

for all formulas $P, Q \in F_{\mathcal{T}}$. A formula $P \in F_{\mathcal{T}}$ is a tautology if $e(P) = \mathbf{1}$ for each truth evaluation $e: F_{\mathcal{T}} \to E$.

Notice that semantics of Basic EQ-logic is formed by means of good, non-commutative EQ-algebras.

Lemma 3.8. ([6])

All axioms of the basic EQ-logic are tautologies.

Lemma 3.9. ([6])

The inference rules of basic EQ-logic are sound in the following sense: Let $e: F_{\mathcal{J}} \to E$ be a truth evaluation where E is a support of a good non-commutative EQ-algebra:

(a) If $e(P) = \mathbf{1}$ and $e(P \equiv Q) = \mathbf{1}$ then $e(Q) = \mathbf{1}$.
(b) If $e(Q \equiv R) = \mathbf{1}$ then $e(P[\mathbf{p} := Q] = P[\mathbf{p} := R]) = \mathbf{1}$ for any formula P.

The following is standard procedure due to Lindenbaum and Tarski[5], we now study and introduce the completeness of the basic EQ-logic [6].

Definition 3.12. ([6])

Put

$$P \approx Q \text{ iff } \vdash P \equiv Q, \quad P, Q \in F_{\mathcal{J}} \tag{3.27}$$

The relation " $\approx$ " is an equivalence on $F_{\mathcal{J}}$. Let us denote by $[P]$ an equivalence class of P and put

$$\bar{E} = \{[P] |\, P \in F_{\mathcal{J}}\} \text{ where } [P] = \{Q| \vdash P \equiv Q.$$

Finally, we define

$$\mathbf{1} = [\top]$$

$$[P] \wedge [Q] = [P \boldsymbol{\wedge} Q]$$

$$[P] \otimes [Q] = [P \mathbin{\&} Q]$$

$$[P] \sim [Q] = [P \equiv Q]$$

---

[5] The Lindenbaum–Tarski algebra is the quotient algebra obtained by factoring the algebra of formulas by the congruence relation.

Lemma 3.10. ([6])

The algebra $\bar{\mathcal{E}} = (\bar{E}, \wedge, \otimes, \sim, \mathbf{1})$ is a good non commutative EQ-algebra.

Theorem 3.6. ([6]) (**Soundness**)

The basic EQ-logic is sound.

Theorem 3.7. ([6]) (**Completeness**)

The following is equivalent for every formula P:

(a) $\vdash P$

(b) $e(P) = \mathbf{1}$ for every good non-commutative EQ-algebra $\mathcal{E}$ and a truth evaluation $e: F_{\mathcal{T}} \to E$.

3.3 Prelinear $\mathbf{EQ_\Delta}$-Logic

In this section, we introduce a complete propositional calculus for prelinear EQ_Δ-algebras which is developed in [5]. It is called *prelinear* EQ_Δ-logic.

3.3.1 Syntax of Prelinear $\mathbf{EQ_\Delta}$-Logic

The language of prelinear EQ_Δ-logic is the same as that of the basic EQ-logic extended by the unary connective " $\mathbf{\Delta}$ " and the truth constant " $\perp$ ". Let $F_{\mathcal{T}}$ denote the set of all formulas for the given language $\mathcal{T}$. This logic is defined on the basis of a prelinear EQ_Δ-algebra of truth values. Further definable connectives are

$$P \vee Q := ((P \Rightarrow Q) \Rightarrow Q) \wedge ((Q \Rightarrow P) \Rightarrow P) \tag{3.28}$$

$$\neg P := P \Rightarrow \perp \tag{3.29}$$

3.3.2 Logical Axioms and Inference Rules

The logical axioms of the prelinear EQ_Δ-logic are the logical axioms (A1), (A2),…, (A11) of the basic EQ-logic plus the following ones:

(A12) $(P \wedge \bot) \equiv \bot$

(A**Δ**0) $\boldsymbol{\Delta}\top$

(A**Δ**1) $\boldsymbol{\Delta}P \Rightarrow \boldsymbol{\Delta}\boldsymbol{\Delta}P$

(A**Δ**2) $\boldsymbol{\Delta}(P \Rightarrow Q) \Rightarrow (\boldsymbol{\Delta}P \Rightarrow \boldsymbol{\Delta}Q)$

(A**Δ**3) $(\boldsymbol{\Delta}(P \Rightarrow Q) \Rightarrow R) \Rightarrow ((\boldsymbol{\Delta}(Q \Rightarrow P) \Rightarrow R) \Rightarrow R)$

(A**Δ**4) $(\boldsymbol{\Delta}P \Rightarrow \neg\boldsymbol{\Delta}P) \Rightarrow \neg\boldsymbol{\Delta}P$

(A**Δ**5) $(\neg\boldsymbol{\Delta}P \Rightarrow \boldsymbol{\Delta}P) \Rightarrow \boldsymbol{\Delta}P$

(A**Δ**6) $\boldsymbol{\Delta}Q \Rightarrow (T \Rightarrow (T \,\&\, (R \Rightarrow (Q \,\&\, R))))$

Inference rules of the prelinear EQ_Δ-logic are the same as that of the basic EQ-logic, i.e. they are equanimity rule (EA) and Leibniz rule (Leib).

The theorems and inferences of the basic EQ-logic remain valid in extension of the prelinear EQ_Δ-logic, since the prelinear EQ_Δ-logic is an extension of the basic EQ-logic.

3.3.3 Semantics of Prelinear EQ_Δ-Logic

It's been explained that the semantical domain for the prelinear EQ_Δ-logic is the class of all prelinear EQ_Δ-algebras. In this section, we introduce the general and chain completeness of prelinear EQ_Δ-logic for the variety of prelinear EQ_Δ-algebras which have been established in [5], i.e. completeness of the whole variety and the class of chains of the variety, respectively.

Definition 3.13. ([5])

Interpretation of the prelinear EQ_Δ-logic is a tuple $\mathfrak{R} = (\boldsymbol{\mathcal{E}}, e)$ in which $\boldsymbol{\mathcal{E}} = (E,\ \wedge,\ \otimes,\ \sim,\ \Delta, \mathbf{0}, \mathbf{1})$ is a prelinear EQ_Δ-algebra and a function $e\colon F_{\mathcal{T}} \to E$, called the *truth evaluation* of the interpretation that satisfies the following identities for all formulas $P, Q \in F_{\mathcal{T}}$:

$$e(\top) = \mathbf{1}, \quad e(\bot) = \mathbf{0},$$
$$e(P \wedge Q) = e(P) \wedge e(Q),$$
$$e(P \mathbin{\&} Q) = e(P) \otimes e(Q),$$
$$e(P \equiv Q) = e(P) \sim e(Q),$$
$$e(\mathbf{\Delta} P) = \Delta e(P).$$

If $e(P) = \mathbf{1}$ in an interpretation $\mathfrak{R}$ then P is said to be *valid* (or, *true*) in $\mathfrak{R}$, and we write $\mathfrak{R} \vDash P$.

Let T be a theory and $\mathfrak{R} = (\boldsymbol{\mathcal{E}}, e)$ be an interpretation, then

If $\mathfrak{R} \vDash P$ for all $P \in T$, we write $\mathfrak{R} \vDash T$,

and we say that $\mathfrak{R}$ is a $\boldsymbol{\mathcal{E}}$-*model* of T.

If for every interpretation $\mathfrak{R}$ such that $\mathfrak{R} \vDash T$ we have $\mathfrak{R} \vDash P$, then we write $T \vDash P$. If $\mathfrak{R} \vDash P$ for all the interpretations $\mathfrak{R}, P$ is called *universally valid* (or, a *tautology*), and we write $\vDash P$.

The following is standard procedure due to Lindenbaum and Tarski, we now study and introduce the completeness of the prelinear $\mathrm{EQ}_{\mathbf{\Delta}}$-logic.

Let T be a theory over the prelinear $\mathrm{EQ}_{\mathbf{\Delta}}$-logic. Then consider the relation (3.27). We explain that (3.27) is an equivalence relation on $F_{\mathcal{T}}$.

Let $\rho: F_{\mathcal{T}} \to F_{\mathcal{T}}/\approx$ be the quotient map onto the set of all equivalence classes $|P| = \{Q \mid T \vdash P \equiv Q\}$. The Leibniz rule (Leib) guarantees that the logical connectives possess the substitution property for " $\approx$ ". In consequence, the following operations are well-defined on the set $\bar{E} = \{|P| \mid P \in F_{\mathcal{T}}\}$:

$$|P| \wedge_T |Q| = q(P \boldsymbol{\wedge} Q),$$
$$|P| \otimes_T |Q| = q(P \boldsymbol{\&} Q),$$
$$|P| \sim_T |Q| = q(P \equiv Q),$$
$$\Delta_T |P| = q(\boldsymbol{\Delta} P).$$

The partial order $\leq$ is also well-defined on $F_{\mathcal{T}}/\approx$ by

$$|P| \leq |Q| \text{ iff } |P| \wedge_T |Q| = |P| \text{ iff } T \vdash (P \wedge Q) \equiv P \\ \text{iff } T \vdash P \Rightarrow Q \tag{3.30}$$

Let $\boldsymbol{\mathcal{E}}_T = (\bar{E}, \wedge_T, \otimes_T, \sim_T, \Delta_T, \mathbf{0}_T, \mathbf{1}_T)$ be the Lindenbaum algebra of the theory T, where $\mathbf{1}_T = \rho(\top)$, $\mathbf{0}_T = \rho(\bot)$. $\boldsymbol{\mathcal{E}}_T$ is a good non-commutative EQ-algebra (see also Lemma 3.10) and the top element $\mathbf{1}_T$ is exactly the equivalence class $\{P \in F_{\mathcal{T}} | T \vdash P\}$. It is bounded (by Axiom (A12)) and its partial order is its lattice order. Hence, by Axioms (AΔ0)-(AΔ6), $\boldsymbol{\mathcal{E}}_T$ is a prelinear EQ_Δ-algebra. Moreover, the quotient map is a truth evaluation. From these arguments with the representation theorem (Theorem 3.5), we conclude the following theorem.

Theorem 3.8. ([5]) (**Completeness**)

The prelinear $\mathrm{EQ}_{\boldsymbol{\Delta}}$-logic is generally complete and chain complete for the variety of prelinear EQ_Δ-algebras. Specifically, for every formula $P \in F_{\mathcal{T}}$ and for every theory T over the prelinear $\mathrm{EQ}_{\boldsymbol{\Delta}}$-logic, the following are equivalent:

(a) $T \vdash P$.

(b) For each prelinear EQ_Δ-algebra $\boldsymbol{\mathcal{E}}$ and each $\boldsymbol{\mathcal{E}}$-model $\mathfrak{R}$ of T, $\mathfrak{R} \models P$.

(c) For each linearly ordered EQ_Δ-algebra $\boldsymbol{\mathcal{E}}$ and each $\boldsymbol{\mathcal{E}}$-model $\mathfrak{R}$ of T, $\mathfrak{R} \models P$.

Theorem 3.9. ([5]) (**Deduction theorem**)

For each theory T, formula P and arbitrary formula Q it holds that:

$$T \cup \{P\} \vdash Q \text{ iff } T \vdash \mathbf{\Delta}P \Rightarrow Q.$$

Chapter 4
$\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-Algebras

In this chapter, we introduce and study a class of separated lattice EQ-algebras that may be represented as subalgebras of products of linearly ordered ones. Such algebras are called representable. Namely, we enrich separated lattice EQ-algebras with a unary operation (the so called Baaz delta), fulfilling some additional assumptions. The resulting algebras are called $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebras. One of the main results of this chapter is to characterize the class of representable $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebras. We also supply a number of useful results, leading to this characterization.

4.1 Definition and Fundamental Properties

Definition 4.1.

A $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebra is an algebra $\mathcal{E}_\Delta = (E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ that is separated ℓEQ-algebra with a bottom element $\mathbf{0}$ expanded by a unary operation $\Delta: E \to E$ fulfilling the following axioms:

(E$_s\Delta$1) $\Delta\mathbf{1} = \mathbf{1}$;

(E$_s\Delta$2) $\Delta p \leq p$;

(E$_s\Delta$3) $\Delta p \leq \Delta\Delta p$;

(E$_s\Delta$4) $\Delta(p \sim q) \leq \Delta p \sim \Delta q$;

(E$_s\Delta$5) $\Delta(p \wedge q) = \Delta p \wedge \Delta q$;

(E$_s\Delta$6) $\Delta(p \vee q) \leq \Delta p \vee \Delta q$;

(E$_s\Delta$7) $\Delta p \vee \neg\Delta p = \mathbf{1}$;

(E$_s\Delta$8) $\Delta(p \sim q) \leq (p \otimes r) \sim (q \otimes r)$;

(E$_s\Delta$9) $\Delta(p \sim q) \leq (r \otimes p) \sim (r \otimes q)$.

Remark 4.1.

The axioms ($E_s\Delta1$), ($E_s\Delta2$),…,($E_s\Delta7$) are from [8] (see Definition 3.8) and the two inequalities ($E_s\Delta8$) and ($E_s\Delta9$) are from [5] (see section 3.1.4) . They are necessary to assure good behavior of the multiplication " $\otimes$ " with respect to the crisp equality. If we omit "Δ" in ($E_s\Delta8$) and ($E_s\Delta9$) then the resulting EQ-algebra becomes residuated.

Lemma 4.1.

Let $\mathcal{E}_\Delta$ be a ℓEQ_Δ^s-algebra. For all $p, q, r \in E$, it holds that:

(a) If $p \leq q$, then $\Delta p \leq \Delta q$;
(b) $\Delta(p \to q) \leq \Delta p \to \Delta q$;
(c) $\Delta(p \vee q) = \Delta p \vee \Delta q$;
(d) $\Delta\Delta p = \Delta p$;
(e) $p \otimes \Delta(p \to q) \leq q,\ \Delta(p \to q) \otimes p \leq q$;
(f) $p \otimes \Delta(p \sim q) \leq q,\ \Delta(p \sim q) \otimes p \leq q$;
(g) $\Delta(p \sim \mathbf{1}) = \Delta p$, and $\Delta(\mathbf{1} \to p) = \Delta p$;
(h) $\Delta q \leq r \to (q \otimes r)$, and $\Delta q \leq r \to (r \otimes q)$;
(i) $\Delta p = \Delta p \otimes \Delta p$;
(j) $\Delta p \leq \Delta q \to \Delta r$ iff $\Delta p \otimes \Delta q \leq \Delta r$ and $\Delta q \otimes \Delta p \leq \Delta r$;
(k) If $\mathcal{E}_\Delta$ is prelinear, then $\Delta(p \to q) \vee \Delta(q \to p) = \mathbf{1}$;
(l) $\Delta(p \to q) \leq (p \otimes r) \to (q \otimes r)$, and $\Delta(p \to q) \leq (r \otimes p) \to (r \otimes q)$.

Proof.

(a): Assume $p \leq q$ ($p \wedge q = p$). Hence, by ($E_s\Delta5$), we have

$$\Delta(p \wedge q) = \Delta p \wedge \Delta q = \Delta p;\ \text{that is}\ \Delta p \leq \Delta q.$$

(b): From ($E_s\Delta4$) and ($E_s\Delta5$), we get

$$\Delta(p \to q) = \Delta((p \wedge q) \sim p) \leq \Delta(p \wedge q) \sim \Delta q = (\Delta p \wedge \Delta q) \sim \Delta q$$
$$= \Delta p \to \Delta q.$$

(c): From item (a) (because $p, q \leq p \vee q$), we can have, $\Delta p, \Delta q \leq \Delta(p \vee q)$. Therefore, $\Delta p \vee \Delta q \leq \Delta(p \vee q)$. Hence, by this and ($E_s\Delta6$), the result holds.

(d): Direct from ($E_s\Delta2$) with item (a), we obtain $\Delta\Delta p \leq \Delta p$. Hence, by this and ($E_s\Delta3$), the result holds.

(e): From ($E_s\Delta2$), Lemma 3.1(g) and the order properties of " $\to$ ", we get

$$\Delta(p \to q) \leq (p \to q) \leq (p \otimes \Delta(p \to q)) \to q,$$

$$\neg\Delta(p \to q) = \Delta(p \to q) \to \mathbf{0} \leq \Delta(p \to q) \to q \leq (p \otimes \Delta(p \to q)) \to q$$

(since $\mathbf{0} \leq q$). Thus, by ($E_s\Delta7$) and Proposition 3.1,

$$(p \otimes \Delta(p \to q)) \to q = \mathbf{1}; \text{ that is } (p \otimes \Delta(p \to q)) \leq q.$$

Similarly, $\Delta(p \to q) \otimes p \leq q$.

(f): Directly from item (e) by Lemma 3.1(h).

(g): By item (d), ($E_s\Delta4$) and item (f), we get

$$\Delta(p \sim \mathbf{1}) = \Delta\Delta(p \sim \mathbf{1}) = \mathbf{1} \otimes \Delta\Delta(\mathbf{1} \sim p) \leq \Delta\mathbf{1} \otimes \Delta(\Delta\mathbf{1} \sim \Delta p) \leq \Delta p.$$

On the other hand, $\Delta p \leq \Delta(p \sim \mathbf{1})$ by item (a) (since $p \leq (p \sim \mathbf{1})$).

In particular, $\Delta(\mathbf{1} \to p) = \Delta((\mathbf{1} \wedge p) \sim \mathbf{1}) = \Delta(p \sim \mathbf{1}) = \Delta p$.

(h): From item (g), ($E_s\Delta8$) and Lemma 3.1(h), we get

$$\Delta q = \Delta(\mathbf{1} \sim q) \leq (\mathbf{1} \otimes r) \sim (q \otimes r) \leq (\mathbf{1} \otimes r) \to (q \otimes r)$$
$$= r \to (q \otimes r).$$

Similarly, $\Delta q \leq r \to (r \otimes q)$.

(i): By item (h), item (d) and order properties of " $\rightarrow$ ", we obtain

$$\Delta p = \Delta\Delta p \leq \Delta p \rightarrow (\Delta p \otimes \Delta p) \text{ and}$$

$$\neg\Delta p = \Delta p \rightarrow \mathbf{0} \leq \Delta p \rightarrow (\Delta p \otimes \Delta p)$$

(since $\mathbf{0} \leq (\Delta p \otimes \Delta p)$). Thus, by ($E_s\Delta7$) and Proposition 3.1, $\Delta p \rightarrow (\Delta p \otimes \Delta p) = \mathbf{1}$; that is $\Delta p \leq (\Delta p \otimes \Delta p)$. On the other hand, $(\Delta p \otimes \Delta p) \leq \Delta p$ by Lemma 3.1(g).

(j): Assume $\Delta p \leq \Delta q \rightarrow \Delta r$, then by Lemma 3.1(g) and the order properties of " $\rightarrow$ ",

$$\Delta p \leq \Delta q \rightarrow \Delta r \leq (\Delta p \otimes \Delta q) \rightarrow \Delta r \text{ and}$$

$$\neg\Delta p = \Delta p \rightarrow \mathbf{0} \leq \Delta p \rightarrow \Delta r \leq (\Delta p \otimes \Delta q) \rightarrow \Delta r.$$

Thus, by ($E_s\Delta7$), and Proposition 3.1, $(\Delta p \otimes \Delta q) \rightarrow \Delta r = \mathbf{1}$; that is $(\Delta p \otimes \Delta q) \leq \Delta r$. Similarly, $(\Delta q \otimes \Delta p) \leq \Delta r$. Conversely, assume $(\Delta p \otimes \Delta q) \leq \Delta r$. Hence, by item (d) and item (h), we obtain

$$\Delta p = \Delta\Delta p \leq \Delta q \rightarrow (\Delta p \otimes \Delta q) \leq \Delta p \rightarrow \Delta r.$$

Similarly, for $(\Delta q \otimes \Delta p) \leq \Delta r$.

(k): By ($E_s\Delta1$), the prelinearity and item (c), we get

$$\mathbf{1} = \Delta\mathbf{1} = \Delta((p \rightarrow q) \vee (q \rightarrow p)) = \Delta(p \rightarrow q) \vee \Delta(q \rightarrow p).$$

(l): Using ($E_s\Delta8$) and the order properties of " $\rightarrow$ ", we have

$$\begin{aligned}\Delta(p \rightarrow q) = \Delta((p \wedge q) \sim p) &\leq ((p \wedge q) \otimes r) \sim (p \otimes r)\\ &\leq (p \otimes r) \rightarrow ((p \wedge q) \otimes r)\\ &\leq (p \otimes r) \rightarrow (q \otimes r).\end{aligned}$$

Similarly, $\Delta(p \rightarrow q) \leq (r \otimes p) \rightarrow (r \otimes q)$. ∎

Theorem 4.1.

The class of $\ell\mathrm{EQ}_\Delta^s$-algebras is a variety.

Proof.

Just note that the separateness axiom (i.e., $p \leq q$ iff $p \to q = \mathbf{1}$ for all $p, q \in E$) is equivalent to the identity $p \otimes \Delta(p \to q) \leq q$, this can be seen as follows: Assume $p \otimes \Delta(p \to q) \leq q$ and let $p \to q = \mathbf{1}$, then

$$p = p \otimes \mathbf{1} = p \otimes \Delta\mathbf{1} = p \otimes \Delta(p \to q) \leq q.$$

Hence, by Lemma 4.1(e) the result holds. Note that we have $p \leq q$ iff $p \wedge q = p$. Hence, all the other properties stated in Definition 3.3 and Definition 4.1 can be expressed using equations (see Theorem 3.1). ■

4.2 Filters in $\ell\mathrm{EQ}_\Delta^s$-algebras

Definition 4.2.

Let $\mathcal{E}_\Delta = (E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ be a $\ell\mathrm{EQ}_\Delta^s$-algebra. A subset $F \subseteq E$ is called a *filter* of $\mathcal{E}_\Delta$ if for all $p, q \in E$:

(a) $\mathbf{1} \in F$.

(b) if $p, p \to q \in F$, then $q \in F$.

(c) if $p \in F$, then $\Delta p \in F$.

Remark 4.2.

A (prime) filer F on a $\ell\mathrm{EQ}_\Delta^s$-algebra $\mathcal{E}_\Delta = (E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ is a (prime) prefilter (in the sense of given in [8]) on its separated EQ-algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, \mathbf{1})$ satisfying (c) (see Definition 3.5). So all the properties of (prime) prefilters on it separated EQ-algebra (see [7, 8]) are also properties of (prime) filters on a $\ell\mathrm{EQ}_\Delta^s$-algebra, including the following result:

Lemma 4.2.

Let F be a filter of a $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebra $\mathcal{E}_\Delta$. For all $p, q \in E$ it holds that:

(a) If $p \in F$ and $p \leq q$ then $q \in F$;
(b) If $p, p \sim q \in F$ then $q \in F$;
(c) If $p, q \in F$ then $p \wedge q \in F$.

Proof.

(a) From Lemma 3.1(e), it follows that $p \to q = \mathbf{1}$. The properties (a) and (b) in Definition 3.5 of a prefilter imply that $p \to q \in F$ and then $q \in F$.

(b) Due to Lemma 3.1(h), it holds that $p \sim q \leq p \to q$. From item (a), it then follows that $p \to q \in F$, so the property (b) in Definition 3.5 of a prefilter implies that $q \in F$.

(c) From Lemma 3.1(j) and Lemma 3.1(n), it follows that $q \leq p \to q = p \to p \wedge q$. From item (a), it then follows that $p \to p \wedge q$ and hence, by the property (b) in Definition 3.5 of a prefilter, $p \wedge q \in F$. ∎

Lemma 4.3.

Let F be a filter of a $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebra $\mathcal{E}_\Delta$. For all $p, q, r, p', q' \in E$ such that $p \sim q \in F$ and $p' \sim q' \in F$, it holds that

(a) If $p \to q \in F$, then $(p \otimes r) \to (q \otimes r) \in F$ and $(r \otimes p) \to (r \otimes q) \in F$
(b) If $p,\ q \in F$ then $p \otimes q \in F$;
(c) $(p \otimes p') \sim (q \otimes q') \in F$ and $(p' \otimes p) \sim (q' \otimes q) \in F$;
(d) $(\Delta p \sim \Delta q) \in F$.

Proof.

(a): Assume $p \to q \in F$. Since F is a filter, then $\Delta(p \to q) \in F$. Hence, by Lemma 4.1(l) and Lemma 4.2(a), we get

$$\Delta(p \to q) \leq (p \otimes r) \to (q \otimes r) \in F.$$

Similarly, $(r \otimes p) \to (r \otimes q) \in F$.

(b): From Lemma 3.1(j) and Lemma 4.2(a), it follows that $q \leq \mathbf{1} \to q \in F$. From item (a), it then follows that

$$(p \otimes \mathbf{1}) \to (p \otimes q) = p \to (p \otimes q) \in F.$$

Hence, by Definition 4.2 of a filter, $p \otimes q \in F$.

(c): By Definition 4.2, $\Delta(p \sim q)$ and $\Delta(p' \sim q') \in F$. Thus, by ($E_s\Delta 8$) and ($E_s\Delta 9$), we get

$$\begin{aligned} \Delta(p \sim q) \otimes \Delta(p' \sim q') &\leq \\ &\leq ((p \otimes p') \sim (q \otimes p')) \otimes ((q \otimes p') \sim (q \otimes q')) \\ &\leq (p \otimes p') \sim (q \otimes q') \end{aligned}$$

Hence, by Lemma 4.2(a) and item (b), the result holds. Similarly, $(p' \otimes p) \sim (q' \otimes q) \in F$.

(d): By Definition 4.2 and Lemma 4.2(a)

$$\Delta(p \sim q) \in F \text{ implies } \Delta p \sim \Delta q \in F \text{ (since } \Delta(p \sim q) \leq \Delta p \sim \Delta q). \quad \blacksquare$$

Lemma 4.4.

Let $\mathcal{E}_\Delta$ be a ℓEQ^s_Δ-algebra. Given a filter $F \subseteq E$, the following relation on $\mathcal{E}_\Delta$ is a congruence relation:

$$p \approx_F q \text{ iff } p \sim q \in F \quad (4.1)$$

Proof.

Indeed, Definition 3.3(E3), Lemma 3.1(a) and Lemma 3.1(b) guarantee that $\approx_F$ is an equivalence relation. As an immediate consequence of Lemma 4.3, all the operations of $\mathcal{E}_\Delta$ are compatible with the relation given by (4.1); that is

$$p \approx_F q \text{ and } p' \approx_F q' \text{ imply } (p \wedge p') \approx_F (q \wedge q'), (p \vee p') \approx_F (q \vee q'),$$
$$(p \sim p') \approx_F (q \sim q'), (p \otimes p') \approx_F (q \otimes q'), \text{ and } (\Delta p \approx_F \Delta q).$$

Then, $\approx_F$ is a congruence relation. ■

Let $\mathcal{E}_\Delta$ be a $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebra. For $p \in E$, we denote its equivalence class with respect to $\approx_F$ by $[p]_F$ and by E/F the quotient set associated with $\approx_F$. Furthermore, we define the factor algebra

$$\mathcal{E}_\Delta/F = \langle E/F,\ \wedge_F,\ \vee_F,\ \otimes_F, \sim_F, \Delta_F, \mathbf{0}_F, \mathbf{1}_F \rangle.$$

in the standard way as follows:

$E/F = \{[p]_F |\ p \in E\}$, and the binary operations on E/F are defined by

$$[p]_F \wedge_F [q]_F = [p \wedge q]_F;$$
$$[p]_F \vee_F [q]_F = [p \vee q]_F;$$
$$[p]_F \sim_F [q]_F = [p \sim q]_F;$$
$$[p]_F \otimes_F [q]_F = [p \otimes q]_F;$$
$$\Delta_F[p]_F = [\Delta p]_F.$$

The top and the bottom elements are $\mathbf{1}_F = [\mathbf{1}]_F = \{q \in E \mid q \sim \mathbf{1} \in F\} = F$), $\mathbf{0}_F = [\mathbf{0}]_F = \mathbf{0}$, respectively.

Also, we can define a binary relation " $\leq_F$ " on E/F as follows:

$$[p]_F \leq_F [q]_F \text{ iff } [p]_F \wedge_F [q]_F = [p]_F \text{ iff } p \wedge q \approx_F p \quad (4.2)$$
$$\text{iff } p \to q \in F$$

Then, we have the following result.

Theorem 4.2.

Let F be a filter of a $\ell\mathrm{EQ}_\Delta^s$-algebra $\mathcal{E}_\Delta$. The factor algebra $\mathcal{E}_\Delta/F = \langle E/F, \wedge_F, \vee_F, \otimes_F, \sim_F, \Delta_F, \mathbf{0}_F, \mathbf{1}_F \rangle$ is a $\ell\mathrm{EQ}_\Delta^s$-algebra, and the mapping $f: E \longrightarrow E/F$ defined by $f(p) = [p]_F$ is a homomorphism of $\mathcal{E}_\Delta$.

Proof.

We first need to verify that $\mathcal{E}_\Delta/F$ fulfills axioms (E1)-(E7) (see Definition 3.3). Using the definition of the factor algebra and its operations above with the axioms (E1)-(E7), we get

Axioms (E1) and (E2) are obvious. We demonstrate for instance the isotonicity of " $\otimes$ ". Let $[p]_F \leq_F [q]_F$ and $[r] \in E/F$. Then $p \to q \in F$ and therefore, $p \otimes r \to q \otimes r \in F$. Hence, $[p]_F \otimes_F [r]_F \leq_F [q]_F \otimes_F [r]_F$.

(E3): By definition $[p] \sim_F [p]_F = [p \sim p]_F = [\mathbf{1}]_F$.
(E4): Axiom (E4) in $\mathcal{E}_\Delta$ states that $((p \wedge q) \sim r) \otimes (s \sim p) \leq r \sim (s \wedge q)$, and then

$$(((p \wedge q) \sim r) \otimes (s \sim p)) \to (r \sim (s \wedge q) \ = \mathbf{1} \in F.$$

Hence, $[((p \wedge q) \sim r) \otimes (s \sim p)] \leq_F \ [r \sim (s \wedge q)]$ or equivalently,

$$(([p] \wedge_F [q]) \sim_F [r]) \otimes_F ([s] \sim_F [p]) \leq_F [r] \sim_F ([s] \wedge_F [q]).$$

Axioms (E5)-(E7) can be shown in a similar way.

Separateness: let $[p]_F \sim_F [q]_F = [\mathbf{1}]_F$, then $[p \sim q]_F = [\mathbf{1}]_F$; that is $p \sim q \in F$. This means that $p \approx_F q$ and hence $[p] = [q]$.

It is sufficient to verify that the other axioms of $\ell\mathrm{EQ}^{\mathrm{s}}_{\Delta}$-algebra hold also in the factor algebra $\mathcal{E}_{\Delta}/F$:

Using the axioms ($\mathrm{E_s}\Delta$1)-($\mathrm{E_s}\Delta$10), we get

($\mathrm{E}_s\Delta$1): $\Delta_F[\mathbf{1}]_F = [\Delta\mathbf{1}]_F = [\mathbf{1}]_F$.

($\mathrm{E}_s\Delta$2): If $\Delta p \leq p$, then $\Delta p \to p = \mathbf{1} \in F$. Hence, $[\Delta p]_F \leq_F [p]_F$; that is $\Delta_F[p]_F \leq_F [p]_F$.

($\mathrm{E}_s\Delta$3): If $\Delta p \leq \Delta\Delta p$, then $\Delta p \to \Delta\Delta p = \mathbf{1} \in F$. Hence, $[\Delta p]_F \leq_F [\Delta\Delta p]_F$ that is; $\Delta_F[p]_F \leq_F \Delta_F\Delta_F[p]_F$.

($\mathrm{E}_s\Delta$4): If $\Delta(p \sim q) \leq \Delta p \sim \Delta q$, then $\Delta(p \sim q) \to (\Delta p \sim \Delta q) = \mathbf{1} \in F$. Hence, $[\Delta(p \sim q)]_F \leq_F [\Delta p \sim \Delta q]_F$; that is

$$\Delta_F([p]_F \sim_F [q]_F) \leq_F \Delta_F[p]_F \sim_F \Delta_F[q]_F.$$

($\mathrm{E}_s\Delta$5):

$$\Delta_F([p]_F \wedge_F [q]_F) = [\Delta(p \wedge q)]_F = [\Delta p \wedge \Delta q]_F = \Delta_F[p]_F \wedge_F \Delta_F[q]_F.$$

($\mathrm{E}_s\Delta$6): If $\Delta(p \vee q) \leq \Delta p \vee \Delta q$, then $\Delta(p \vee q) \to \Delta p \vee \Delta q = \mathbf{1} \in F$. Hence, $[\Delta(p \vee q)]_F \leq_F [\Delta p \vee \Delta q]_F$; that is

$$\Delta_F([p]_F \vee_F [q]_F) \leq_F \Delta_F[p]_F \vee_F \Delta_F[q]_F.$$

($\mathrm{E}_s\Delta$7): $\Delta_F[p]_F \vee_F \neg\Delta_F[p]_F = [\Delta p \vee \neg\Delta p]_F = [\mathbf{1}]_F = F$.

($\mathrm{E}_s\Delta$8): If $\Delta(p \sim q) \leq (p \otimes r) \sim (q \otimes r)$, then $\Delta(p \sim q) \to (p \otimes r) \sim (q \otimes r) = \mathbf{1} \in F$. Hence, $[\Delta(p \sim q)]_F \leq_F [(p \otimes r) \sim (q \otimes r)]_F$; that is

$$\Delta_F([p]_F \sim_F [q]_F) \leq_F ([p]_F \otimes_F [r]_F) \sim_F ([q]_F \otimes_F [r]_F).$$

Similarly, ($\mathrm{E_s}\Delta$9).

($\mathrm{E}_s\Delta$10): $([p]_F \to_F [q]_F) \vee_F ([q]_F \to_F [p]_F) = [(p \to q) \vee (q \to p)]_F$

$$= [\mathbf{1}]_F$$

Finally, f is a homomorphism by definition:$f(p \boxdot q) = [p \boxdot q]_F =$

$$f(p \boxdot q) = [p \boxdot q]_F = [p]_F \boxdot_F [q]_F = f(p) \boxdot_F f(q)$$

where $\boxdot \in \{\wedge, \vee, \otimes, \sim\}$ and $f(\Delta p) = [\Delta p]_F = \Delta_F [p]_F = \Delta_F f(p)$. ■

The collection of all filters of a $\ell \mathrm{EQ}_\Delta^s$-algebra $\mathcal{E}_\Delta$will be denoted by $\mathcal{F}(\mathcal{E}_\Delta)$.

4.3 Representable $\ell \mathbf{EQ}_\Delta^s$-algebras

For a nonempty subset X of a $\ell \mathrm{EQ}_\Delta^s$-algebra $\mathcal{E}_\Delta$, the smallest filter of $\mathcal{E}_\Delta$ which contains X, i.e. $\bigcap \{F \in \mathcal{F}(\mathcal{E}_\Delta): X \subseteq F\}$ is said to be a filter of $\mathcal{E}_\Delta$ generated by X and will be denoted by $\langle X \rangle$. It is clear that if $X_1 \subseteq X_2$, then $\langle X_1 \rangle \subseteq \langle X_2 \rangle$. If $X = Y \cup \{p\}$, we will write $\langle Y, p \rangle$ for $\langle X \rangle$. The set of non-negative integers will be denoted by ω, we define

$$p \to^0 q = q, p \to^{n+1} q = p \to (p \to^n q).$$

If $p = 1$, $p \to^{n+1} q$ is denoted by $\tilde{q}^{n+1}$.

The following theorem gives a characterization of a filter generated by a set.

Theorem 4.3.

Let X be a nonempty subset of a $\ell \mathrm{EQ}_\Delta^s$-algebra $\mathcal{E}_\Delta$. Then

$$\langle X \rangle = \{p \in E: \Delta q_1 \to (\Delta q_2 \to \cdots (\Delta q_n \to p) \ldots)) = \mathbf{1},$$
$$\text{for some } q_i \in X, n \in \omega\}.$$

Proof.

Put $M = \{p \in E: \Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q_n \to p) \ldots)) = \mathbf{1}$, for some $q_i \in X, n \in \omega\}$. Now, we show that M is a filter of $\mathcal{E}_\Delta$. Since all $q_i \in M, q_i \leq \mathbf{1}$, therefore by Lemma 4.1(a) and ($\mathrm{E_s}\Delta 1$) $\Delta q_i \leq \Delta \mathbf{1} = \mathbf{1}$ so $\Delta q_i \to \mathbf{1} = \mathbf{1}$; i.e., $\mathbf{1} \in M$. Now, let $p, p \to q \in M$, then there exist $q_1, q_2, \ldots, q_n, q_1', q_2', \ldots, q_m' \in X$ such that

$$\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q_n \to p)\ldots)) = \mathbf{1} \text{ and}$$

$$\Delta q_1' \to (\Delta q_2' \to \ldots (\Delta q_m' \to (p \to q))\ldots)) = \mathbf{1}$$

Hence, by Lemma 3.1(l), we have:

$$\begin{aligned} p \to q &\leq (\Delta q_n \to p) \to (\Delta q_n \to q) \\ &\leq (\Delta q_{n-1} \to (\Delta q_n \to p)) \to (\Delta q_{n-1} \to (\Delta q_n \to q)). \end{aligned}$$

By continuing this way, we get that

$$p \to q \leq$$
$$\leq (\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q_n \to p)\ldots)) \to (\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q_n \to q)\ldots)).$$

Then, by order properties of " $\to$ ", Lemma 4.1(a) and ($E_s\Delta 1$), we conclude that

$$\begin{aligned} p \to q &\leq \mathbf{1} \to (\Delta q_1 \to (\Delta q_2 \to \cdots (\Delta q_n \to q) \ldots)) \\ &\leq \Delta q_0 \to (\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q_n \to q)\ldots)), \end{aligned}$$

where $q_0 \in M$. Hence,

$$\Delta q_m' \to (p \to q) \leq \Delta q_m' \to \Delta q_0 \to ((\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q_n \to q)\ldots))).$$

We can obtain by continuing

$$\Delta q_1' \to (\Delta q_2' \to \ldots (\Delta q_m' \to (p \to q))\ldots) \leq \Delta q_1' \to (\Delta q_2' \to \ldots (\Delta q_m' \to (\Delta q_0 \to (\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q_n \to q)\ldots)))\ldots).$$

Then,

$$\Delta q_1' \to (\Delta q_2' \to \ldots (\Delta q_m' \to (\Delta q_0 \to (\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q_n \to q)\ldots)))\ldots) = \mathbf{1}.$$

And so $q \in M$. Finally, we will prove that $\Delta p \in M$ whenever $p \in M$. Assume that $p \in M$, then

$$(\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q \to p)\ldots)) = \mathbf{1} \text{ for some } q_1, q_2, \ldots, q_n \in X.$$

By ($E_s\Delta 1$), Lemma 4.1(b), Lemma 4.1(d), and the order properties of "$\to$",

$$\begin{aligned}\mathbf{1} = \Delta\mathbf{1} &= \Delta(\Delta q_1 \to (\Delta q_2 \to \cdots (\Delta q_n \to p) \ldots)) \\ &\le (\Delta\Delta q_1 \to (\Delta\Delta q_2 \to \cdots (\Delta\Delta q_n \to \Delta p) \ldots)) \\ &= (\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q \to \Delta p)\ldots)).\end{aligned}$$

Hence, $\Delta p \in M$. Therefore, M is a filter of $\mathcal{E}_\Delta$. Let $F \in \mathcal{F}(\mathcal{E}_\Delta), X \subseteq F$ and $p \in M$, then

$$(\Delta q_1 \to (\Delta q_2 \to \ldots (\Delta q_n \to p)\ldots)) = \mathbf{1}, \text{ for some } q_i \in X \text{ and } n \in \omega.$$

Since $\mathbf{1}, \Delta q_1, \Delta q_2, \ldots, \Delta q_n \in F$, we imply $p \in F$. Thus, $M \subseteq F$. Therefore, M is the smallest filter of $\mathcal{E}_\Delta$ containing X. i.e. $M = \langle X \rangle$. ∎

Theorem 4.4.

Let F be a filter of a $\ell \mathrm{EQ}_\Delta^s$-algebra $\mathcal{E}_\Delta$. Then

$$\langle F, p \rangle = \{q \in E : \Delta p \to q \in F\}$$

Proof.

Let $q \in \langle F, p \rangle$, then by Theorem 4.3 and Lemma 3.1(o) for some $f_1, f_2, \ldots, f_n \in F, n, k_1, k_2 \in \omega$

$$\Delta f_1 \to (\Delta f_2 \to \ldots (\Delta f_n \to (\Delta p \to^{k_1} \tilde{q}^{k_2})\ldots) = \mathbf{1}.$$

Since F is a filter and $\mathbf{1} \in F$, then $\Delta p \to^{k_1} \tilde{q}^{k_2} \in F$. Hence, by Lemma 3.1 (p) and Lemma 4.1(i) we get,

$$\Delta p \to^{k_1} \tilde{q}^{k_2} \le (\Delta p \otimes \ldots \otimes \Delta p) \to \tilde{q}^{k_3} = \Delta p \to \tilde{q}^{k_3} \in F$$

for some $k_3 \in \omega$. Since F is a filter, then by Lemma 4.1(b), Lemma 4.1(d) and Lemma 4.1(g) and Lemma 4.2(a), we obtain

$$\Delta(\Delta p \to \tilde{q}^{k_3}) \leq \Delta\Delta p \to \Delta\tilde{q}^{k_3} = \Delta p \to \Delta q \leq \Delta p \to q \in F$$

Thus, $q \in \{q \in E: \Delta f \to (\Delta p \to q) = \mathbf{1}$ for some $f \in F\}$.

Conversely, since $\langle F, p\rangle$ is a filter and $p \in \langle F, p\rangle$, then $\Delta p \in \langle F, p\rangle$. If $\Delta p \to q \in F$, then $\Delta p \to q \in \langle F, p\rangle$, and hence, $q \in \langle F, p\rangle$. ■

By the following theorem, we determine filters generated by join of two elements.

Theorem 4.5.

Let F be a filter of a $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebra $\mathcal{E}_\Delta$, and $p, q \in E$. Then

$$p \vee q \in F \text{ implies } \langle F, p\rangle \cap \langle F, q\rangle = F;$$

Proof.

It is clear that $F \subseteq \langle F, p\rangle \cap \langle F, q\rangle$. Let $p \vee q \in F$, then by Definition 4.2 and Lemma 4.1(c), $\Delta(p \vee q) = \Delta p \vee \Delta q \in F$. Now let $r \in \langle F, p\rangle \cap \langle F, q\rangle$, then by Theorem 4.4, we get $\Delta p \to r \in F$ and $\Delta q \to r \in F$ for some $f \in F$. Hence, by Lemma 4.3(b), we have $(\Delta p \to r) \otimes (\Delta q \to r) \in F$. By this, Proposition 3.3(b) and Lemma 4.2(a), we have

$$(\Delta p \to r) \otimes (\Delta q \to r) \leq (\Delta p \vee \Delta q) \to r \in F.$$

Therefore, $r \in F$. Thus, $\langle F, p\rangle \cap \langle F, q\rangle \subseteq F$. ■

We extend to $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebra the following result, proved by El-Zekey in [7]. The proof is completely the same as El-Zekey's proof.

Proposition 4.1.

Let F be a filter of a $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebra $\mathcal{E}_\Delta$. Then the following properties are equivalent:

(a) F is prime.

(b) E/F is a chain, i.e., is linearly (totally) ordered by $\leq_F$.

Proof. ([7])

(a) ⇔ (b): If F is prime, then from (4.2) we get

$$(p \to q) \in F \text{ or } (q \to p) \in F \text{ iff } [p]_F \leq_F [q]_F \text{ or } [q]_F \leq_F [p]_F;$$

that is E/F is a chain. ■

Theorem 4.6.

Let $\mathcal{E}_\Delta$ be a $\ell\mathrm{EQ}_\Delta^s$-algebra and let $p \in E, p \neq \mathbf{1}$. Then, there is a prime filter F on $\mathcal{E}_\Delta$ not containing p.

Proof.

There are filters not containing p, e.g. $F_0 = \{\mathbf{1}\}$. We shall show that if F is any filter not containing p and $x, y \in E$ such that $(x \to y) \notin F$ and $(y \to x) \notin F$, then there is a filter $F' \supseteq F$ not containing p but containing either $(x \to y) \in F$ or $(y \to x) \in F$. Note that the least filter F' containing F as a subset and $u \in E$ as an element is $F' = \{v \in E: \Delta u \to v \in F\}$. Indeed, F' is obviously a filter by Theorem 4.4 equivalently $F' = \langle F, u\rangle$.

Thus, assume $(x \to y) \notin F, (y \to x) \notin F$ and let F_1, F_2 be the smallest filters containing F as a subset and $(x \to y), (y \to x)$ respectively as an element. We claim that $p \notin F_1$ or $p \notin F_2$. Assume the contrary; then,

$$\Delta(x \to y) \to p \in F \text{ and } \Delta(y \to x) \to p \in F.$$

Hence, by Lemma 4.3(b), we have

$$(\Delta(x \to y) \to p) \otimes (\Delta(y \to x) \to p) \in F.$$

By this, Proposition 3.3(b) and Lemma 4.2(a), we have

$$(\Delta(x \to y) \to p) \otimes (\Delta(y \to x) \to p) \le (\Delta(x \to y) \vee \Delta(y \to x)) \to p$$
$$= \mathbf{1} \to p \in F.$$

Thus, $p \in F$ (since $\mathbf{1} \in F$) a contradiction. Hence $p \notin F_1$ or $p \notin F_2$.

Now, if $\mathcal{E}_\Delta$ is countable (which will be our case in the proof of completeness), then we may arrange all pairs (x, y) from E^2 into a sequence $\{(x_n, y_n) | n \text{ natural}\}$, put $F_0 = \{\mathbf{1}\}$ and having constructed F_n such that $p \notin F_n$ we take $F_{n+1} \supseteq F_n$ such that $p \notin F$ according to our construction; if possible we take F_{n+1} such that $(x_n \to y_n) \in F_{n+1}$, if not, we take that with $(y_n \to x_n) \in F_{n+1}$. Our desired prime filter is the union

$$\bigcup_n F_n$$

If $\mathcal{E}_\Delta$ is uncountable, then one has to use the axiom of choice and work similarly with a transfinite sequence of filters. ■

Theorem 4.7. (**Representation theorem**)

Let $\mathcal{E}_\Delta$ be prelinear $\ell\mathrm{EQ}_\Delta^s$-algebra. Then, each $\mathcal{E}_\Delta$ is subdirectly embeddable into a product of linearly ordered $\ell\mathrm{EQ}_\Delta^s$-algebras; i.e., $\mathcal{E}_\Delta$ is representable.

Proof.

Let $\boldsymbol{\mathcal{P}}$ be the set of all prime filters of $\mathcal{E}_\Delta$. For $F \in \boldsymbol{\mathcal{P}}$. Thus, by Theorem 4.2, the natural homomorphism $h: \mathcal{E}_\Delta \to \prod_{F \in \boldsymbol{\mathcal{P}}} \mathcal{E}_\Delta/{\approx_F}$ defined by $h(p) = \langle [p]_F \rangle_{F \in \boldsymbol{\mathcal{P}}}$ is a subdirect embedding of $\mathcal{E}_\Delta$ into a direct product of $\{\mathcal{E}_\Delta/{\approx_F} : F \in \boldsymbol{\mathcal{P}}\}$. It remains to show that it is one-one. If $p, q \in F$ and $p \neq q$ then $p \nleq q$ or $q \nleq p$. Without loss of generality, then $(p \to q) \neq \mathbf{1}$ in E. By Theorem 4.6, let F be a prime filter on E not containing $(p \to q)$; then in $\mathcal{E}_\Delta/F$, $[p]_F \nleq [q]_F$, hence $[p]_F \neq [q]_F$ and therefore $h(p) \neq h(q)$. Using

Proposition 4.1 and Theorem 4.2, $\mathcal{E}_{\Delta}/{\approx_F}$ is linearly ordered $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebra for each $F \in \boldsymbol{\mathcal{P}}$, which completes the proofs. ■

Chapter 5
$\ell\mathrm{EQ}_{\Delta}^{s}$-Logic

In this chapter, we develop many-valued (fuzzy) logic in which the basic connective is fuzzy equality and the implication is derived from the latter. Precisely, we formulate the $\ell\mathrm{EQ}_{\Delta}^{s}$-logic which is rich enough to enjoy the completeness property and its set of truth values is formed by $\ell\mathrm{EQ}_{\Delta}^{s}$-algebras in which the fuzzy equality is one of the basic operations. The implication operation (as well as the corresponding connective) is derived. We in detail introduce syntax and semantics of the $\ell\mathrm{EQ}_{\Delta}^{s}$-logic and prove various theorems characterizing its properties including completeness. Formal proofs in this chapter proceed mostly in an equational style.

5.1 $\ell\mathrm{EQ}_{\Delta}^{s}$-logic: syntax

Definition 5.1.

The language of $\ell\mathrm{EQ}_{\Delta}^{s}$-logic is the language of the basic logic expanded by the binary connective $\vee$, the unary connective $\boldsymbol{\Delta}$ and a false (logical) constant $\bot$. Implication is a derived connective defined by (3.26). Further definable connective is (3.29). The truth constant $\top$ is defined by:

$$\top =: \bot \equiv \bot \tag{5.1}$$

Let $\mathcal{T}$ be a language of $\ell\mathrm{EQ}_{\Delta}^{s}$-logic and the algebra of truth values is formed by $\ell\mathrm{EQ}_{\Delta}^{s}$-algebra $\mathcal{E}_{\Delta} = (E,\ \wedge,\ \vee,\ \otimes, \sim,\ \Delta, \mathbf{0}, \mathbf{1})$.

The set of all formulas for the given language $\mathcal{T}$ is denoted by $F_{\mathcal{T}}$.

5.1.1 Logical Axioms and Inference Rules

The logical axioms of the $\ell\mathrm{EQ}_{\Delta}^{s}$-logic consist of the logical axioms (A2), (A3),…, (A11) of the basic EQ-logic plus the following ones:

(A$_s$1) $P \vee P \equiv P$

(A$_s$2) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

(A$_s$3) $P \square (P \boxdot Q) \equiv P$ where $\{\square, \boxdot\} = \{\wedge, \vee\}, \square \neq \boxdot$

(A$_s$4) $(((P \vee Q) \equiv R) \& (S \equiv P) \Rightarrow (R \equiv (Q \vee S)))$

(A$_s$5) $P \Rightarrow (\top \equiv P)$

(A$_s$6) $P \wedge \bot \equiv \bot$

(A$_s$7) $\boldsymbol{\Delta}\top$

(A$_s$8) $\boldsymbol{\Delta}P \equiv \boldsymbol{\Delta}P \wedge P$

(A$_s$9) $\boldsymbol{\Delta\Delta}P \equiv \boldsymbol{\Delta}P$

(A$_s$10) $\boldsymbol{\Delta}(P \Rightarrow Q) \Rightarrow (\boldsymbol{\Delta}P \Rightarrow \boldsymbol{\Delta}Q)$

(A$_s$11) $\boldsymbol{\Delta}(P \Rightarrow Q) \vee \boldsymbol{\Delta}(Q \Rightarrow P)$

(A$_s$12) $\boldsymbol{\Delta}(P \equiv Q) \Rightarrow ((R \& P) \& S) \equiv (R \& (Q \& S))$

(A$_s$13) $\boldsymbol{\Delta}P \vee \neg \boldsymbol{\Delta}P$

Remark 5.1.

Our aim is developing a more general fuzzy EQ-logic whose semantics based on separateness (need not to be good) called $\ell\mathrm{EQ}_{\boldsymbol{\Delta}}^{\mathrm{s}}$-algebras. Consequently, we formulate the axiom (A$_s$5) as a relaxation from axiom (A1) (goodness axiom) of basic EQ-logic.

Inference rules of $\ell\mathrm{EQ}_{\boldsymbol{\Delta}}^{\mathrm{s}}$-logic are Leibniz rule (Leib) and the Modus Ponens rule (MP):

$$\frac{P,\ P \Rightarrow Q}{Q} \qquad \text{(MP)}$$

5.1.2 Fundamental Properties of $\ell\mathrm{EQ}_{\boldsymbol{\Delta}}^{\mathrm{s}}$-logic

The following lemmas illustrate the main properties of the $\ell\mathrm{EQ}_{\boldsymbol{\Delta}}^{\mathrm{s}}$-logic.

Lemma 5.1.

(a) $P, P \equiv Q \vdash Q$ (Equinimity (EA))

(b) $P \vdash \top \equiv P$; (Rule (T))

(c) $P \vdash \mathbf{\Delta} P$; (Necessitation rule (N))

(d) $\top \equiv P \vdash P$.

Proof.

(a)

$P \equiv Q$ (Assumption)

$\Leftrightarrow \langle (\text{Leib}); \text{"C} - \text{part"}: (P \wedge \mathbf{p}) \rangle$

$P \wedge P \equiv P \wedge Q$

$\Leftrightarrow \langle (\text{Leib}) + (P \wedge P \equiv P); \text{"C} - \text{part"}: (\mathbf{p} \equiv P \wedge Q) \rangle$

$P \equiv P \wedge Q$

That is $\vdash P \Rightarrow Q$. Hence, by (**MP**) with the assumption P, we get $\vdash Q$.

(b) Direct from the assumption and (A_s5) by (**MP**).

(c)

$\top \equiv P$ (Assumption + Item (b))

$\Leftrightarrow \langle (\text{Leib}); \text{"C} - \text{part"}: (\mathbf{\Delta p}) \rangle$

$\mathbf{\Delta} \top \equiv \mathbf{\Delta} P$

Thus, by (**EA**) with (A_s7), we get the result.

(d)

$\top \equiv P$ (Assumption)

$\Leftrightarrow \langle (\text{Leib}); \text{"C} - \text{part"}: (\mathbf{\Delta p}) \rangle$

$\mathbf{\Delta} \top \equiv \mathbf{\Delta} P$

Thus, by (**EA**) with (A_s7), we get $\vdash \mathbf{\Delta}P$. Hence, by (**MP**) with (A_s8) the result holds. ■

Remark 5.2.

The following properties of the basic EQ-logic were proved by the inference rules of ℓEQ_{Δ}^{s}-logic, Equanimity and the logical axioms (A2), (A3),…, (A11) without using goodness axiom (A1) (see [6]). So, they remain valid in the ℓEQ_{Δ}^{s}-logic. We derive further properties in the ℓEQ_{Δ}^{s}-logic that we will use for establishing its completeness for the semantical domain of ℓEQ_{Δ}^{s}-algebras.

Lemma 5.2. ([6])

(a) $P \equiv Q \vdash Q \equiv P$;

(b) $\vdash P \equiv P$;

(c) $P, Q \vdash P \square Q$; where $\square \in \{\&, \wedge, \equiv\}$

(d) $\vdash (P \equiv Q) \equiv (Q \equiv P)$;

(e) $\vdash (P \Rightarrow Q) \Rightarrow ((P \wedge R) \Rightarrow Q)$;

(f) $\vdash (P \equiv Q) \Rightarrow ((P \equiv R) \equiv (Q \equiv R))$;

(g) $\vdash (P \equiv Q) \Rightarrow (P \Rightarrow Q)$;

(h) $P \Rightarrow Q, Q \Rightarrow R \vdash P \Rightarrow R$;

(i) $P \Rightarrow Q, R \Rightarrow S \vdash (P\&R) \Rightarrow (Q\&S)$;

(j) $\vdash (P \equiv Q)\&(Q \equiv R) \Rightarrow (P \equiv R)$;

(k) $\vdash (P\&Q) \Rightarrow P$ and $\vdash (P\&Q) \Rightarrow Q$;

(l) $(P \Rightarrow Q), (P \Rightarrow R) \vdash (P \Rightarrow (Q \wedge R))$;

(m) $\vdash (P \equiv Q) \Rightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$;

(n) $\vdash (P \wedge Q) \Rightarrow P$;

(o) $\vdash (P \equiv Q)\&(R \equiv S) \Rightarrow ((P \equiv R) \equiv (Q \equiv S))$.

Lemma 5.3.

(a) $\vdash P \vee Q \equiv Q \vee P$;

(b) $\vdash P \Rightarrow (P \vee Q)$ and $\vdash Q \Rightarrow (P \vee Q)$;

(c) $\vdash (P \Rightarrow Q)\&(Q \Rightarrow P) \Rightarrow (P \equiv Q)$;

(d) $\vdash (P \Rightarrow Q) \equiv ((P \vee Q) \Rightarrow Q)$;

(e) $P \Rightarrow Q \vdash (R \Rightarrow P) \Rightarrow (R \Rightarrow Q)$;

(f) $P \Rightarrow Q \vdash (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$;

(g) $\vdash (P \Rightarrow Q) \Rightarrow ((P \vee R) \Rightarrow (Q \vee R))$;

(h) $(P \Rightarrow Q), (R \Rightarrow S) \vdash ((P \vee R) \Rightarrow (Q \vee S))$;

(i) $\boldsymbol{\Delta}(P \Rightarrow Q) \Rightarrow R,\ \boldsymbol{\Delta}(Q \Rightarrow P) \Rightarrow R \vdash R$; (Conclusion)

(j) $\vdash (P \Rightarrow Q) \vee (Q \Rightarrow P)$;

(k) $(P \Rightarrow Q) \Rightarrow R,\ (Q \Rightarrow P) \Rightarrow R \vdash R$; (Conclusion)

(l) $\vdash (P \Rightarrow Q) \equiv (P \Rightarrow (P \wedge Q))$;

(m) $\vdash Q \Rightarrow (P \Rightarrow Q)$;

(n) $\vdash P \vee \bot \equiv P$;

(o) $\vdash (P \equiv Q) \Rightarrow ((P \wedge R) \equiv (Q \wedge R))$;

(p) $\vdash (P \equiv Q)\&(R \equiv S) \Rightarrow ((P \wedge R) \equiv (Q \wedge S))$

(q) $\vdash P \equiv Q \Rightarrow (P \vee R) \equiv (Q \vee R)$.

Proof.

(a) From double using of Lemma 5.2(b) and Lemma 5.2(c), we have

$$\vdash (P \vee Q \equiv P \vee Q)\&(P \equiv P).$$

Hence, by (**MP**) with (A_s4) in the form

$$\vdash (P \vee Q \equiv P \vee Q)\&(P \equiv P) \Rightarrow (P \vee Q \equiv Q \vee P),$$

we get $\vdash (P \vee Q \equiv Q \vee P)$.

(b) Using (3.26) with (A_s3) and Lemma 5.2(a), we get $\vdash P \Rightarrow (P \lor Q)$. Hence, by the Leibniz rule (Leib) with item (a) we have the second part.

(c)

$$(P \equiv (P \land Q))\&((P \land Q) \equiv Q) \Rightarrow (P \equiv Q) \qquad \text{(Lemma 5.2(j))}$$
$$\Leftrightarrow \langle \text{(Leib)} + \text{Lemma 5.2(d); "C} - \text{part"}: (P \equiv (P \land Q))\&\mathbf{p} \Rightarrow (P \equiv Q)\rangle$$
$$(P \equiv (P \land Q))\&(Q \equiv (P \land Q)) \Rightarrow (P \equiv Q)$$
$$\Leftrightarrow \langle \text{(Leib)} + \text{(A2); "C} - \text{part"}: (P \equiv (P \land Q))\&(Q \equiv \mathbf{p}) \Rightarrow (P \equiv Q)\rangle$$
$$(P \equiv (P \land Q))\&(Q \equiv (Q \land P)) \Rightarrow (P \equiv Q)$$

(d)

$$(P \lor Q) \equiv ((P \lor Q) \land Q)$$
$$\Leftrightarrow \langle \text{(Leib)} + \text{(A2)} + \text{Item (a); "C} - \text{part"}: (P \lor Q) \equiv \mathbf{p}\rangle$$
$$(P \lor Q) \equiv (Q \land (Q \lor P))$$
$$\Leftrightarrow \langle \text{(Leib)} + (\text{A}_s3)\text{; "C} - \text{part"}: (P \lor Q) \equiv \mathbf{p}\rangle$$
$$(P \lor Q) \equiv Q$$
$$\Leftrightarrow \langle \text{(Leib); "C} - \text{part"}: \mathbf{p} \land P\rangle$$
$$(P \lor Q) \land P \equiv Q \land P$$
$$\Leftrightarrow \langle \text{(Leib)twice} + \text{(A2)}\rangle$$
$$P \land (P \lor Q) \equiv P \land Q$$
$$\Leftrightarrow \langle \text{(Leib)} + (\text{A}_s3)\text{; "C} - \text{part"}: \mathbf{p} \equiv P \land Q\rangle$$
$$P \equiv P \land Q$$

(e)

$$(R \Rightarrow (Q \land P)) \Rightarrow (R \Rightarrow Q) \qquad \text{(A11)}$$
$$\Leftrightarrow \langle \text{(Leib)} + \text{(A2); "C} - \text{part"}: (R \Rightarrow \mathbf{p}) \Rightarrow (R \Rightarrow Q)\rangle$$
$$(R \Rightarrow (P \land Q)) \Rightarrow (R \Rightarrow Q)$$
$$\Leftrightarrow \langle \text{(Leib)} + (P \equiv P \land Q) + \text{Lemma 5.2(a); "C} - \text{part"}:$$

$(R \Rightarrow \mathbf{p}) \Rightarrow (R \Rightarrow Q))\rangle$

$(R \Rightarrow P) \Rightarrow (R \Rightarrow Q)$

(f) In the same way as above using Lemma 5.2(e).

(g) From item (b) and item (e), we get

$\vdash (P \Rightarrow Q) \Rightarrow (P \Rightarrow (Q \vee R)).$

By this, and item (d) using the Leibniz rule, we obtain

$\vdash (P \Rightarrow Q) \Rightarrow ((P \vee (Q \vee R)) \Rightarrow (Q \vee R)).$

From this, and commutativity and associativity of " $\vee$ " using (Leib), we get

$\vdash (P \Rightarrow Q) \Rightarrow (((P \vee R) \vee Q) \Rightarrow (Q \vee R)).$

We can get from item (b) and item (f)

$\vdash (((P \vee R) \vee Q) \Rightarrow (Q \vee R)) \Rightarrow ((P \vee R)) \Rightarrow (Q \vee R)).$

Hence, by Lemma 5.2(h), we get the result.

(h) From the assumptions and item (g) by (**MP**), we have

$\vdash (P \vee R) \Rightarrow (Q \vee R)$ and $(Q \vee R) \Rightarrow (Q \vee S)$.

Lemma 5.2(h) yields the result.

(i) From the assumptions by item (h), we obtain

$\vdash \mathbf{\Delta}(P \Rightarrow Q) \vee \mathbf{\Delta}(Q \Rightarrow P) \Rightarrow (R \vee R).$

From this, and (A_S11) by (**MP**), we obtain $\vdash (R \vee R)$. Hence, by (**EA**) with (A_S1) we get $\vdash R$.

(j) Assuming $\mathbf{\Delta}(P \Rightarrow Q) \Rightarrow (P \Rightarrow Q)$ and $\mathbf{\Delta}(Q \Rightarrow P) \Rightarrow (Q \Rightarrow P)$. Then, by item (h) we get

$\vdash (\mathbf{\Delta}(P \Rightarrow Q) \vee \mathbf{\Delta}(Q \Rightarrow P)) \Rightarrow ((P \Rightarrow Q) \vee (Q \Rightarrow P)).$

Then, by (**MP**) with (A_s11), we obtain the result.

(k) It follows by exactly the similar proof as item (i).

(l)

$P \equiv P \wedge Q$

$\Leftrightarrow \langle$(Leib) + Lemma 5.2(n); "C − part": $P \equiv \mathbf{p}\rangle$

$P \equiv (P \wedge Q) \wedge P$

$\Leftrightarrow \langle$(Leib) + (A1); "C − part": $P \equiv \mathbf{p}\rangle$

$P \equiv P \wedge (P \wedge Q)$ (i.e. $P \Rightarrow (P \wedge Q)$)

(m)

$(\top \equiv \top \wedge Q) \Rightarrow ((\top \wedge P) \Rightarrow Q)$ (Lemma 5.2(e))

$\Leftrightarrow \langle$(Leib)twice + (A2)$\rangle$

$(\top \equiv Q \wedge \top) \Rightarrow ((P \wedge \top) \Rightarrow Q)$

$\Leftrightarrow \langle$(Leib)twice + (A6)$\rangle$

$(\top \equiv Q) \Rightarrow (P \Rightarrow Q)$

Which together with (A_s5) $\vdash Q \Rightarrow (\top \equiv Q)$ yields by Lemma 5.2(h) the formula $\vdash Q \Rightarrow (P \Rightarrow Q)$.

(n)

$P \vee \bot$

$\Leftrightarrow \langle$(Leib) + (A_s6) + Lemma 5.2(a); "C − part": $P \vee \mathbf{p}\rangle$

$P \vee (P \wedge \bot)$

$\Leftrightarrow \langle$(A_s3)$\rangle$

P

(o)

$((P \wedge R) \equiv (P \wedge R))\&(Q \equiv P) \Rightarrow ((P \wedge R) \equiv (Q \wedge R))$ (A9)

$\Leftrightarrow \langle(\text{Leib}) + \text{Lemma 5.2}((\text{a}) + (\text{b})) + \text{Rule (T)}; \text{"C} - \text{part"}:$

$\quad \mathbf{p}\&(Q \equiv P) \Rightarrow ((P \wedge R) \equiv (Q \wedge R))\rangle$

$\top\&(Q \equiv P) \Rightarrow ((P \wedge R) \equiv (Q \wedge R))$

$\Leftrightarrow \langle(\text{Leib}) + (\text{A5}); \text{"C} - \text{part"}: \mathbf{p} \Rightarrow ((P \wedge R) \equiv (Q \wedge R))\rangle$

$(Q \equiv P) \Rightarrow ((P \wedge R) \equiv (Q \wedge R))$

$\Leftrightarrow \langle(\text{Leib}) + \text{Lemma 5.2}(\text{d}); \text{"C} - \text{part"}: \mathbf{p} \Rightarrow ((P \wedge R) \equiv (Q \wedge R))\rangle$

$(P \equiv Q) \Rightarrow ((P \wedge R) \equiv (Q \wedge R))$

(p) From item (o), and Lemma 5.2(i) we get

$$\vdash ((P \equiv Q)\&(R \equiv S)) \Rightarrow ((P \wedge R) \equiv (Q \wedge R)\&(R \wedge Q) \equiv (S \wedge Q)).$$

From this, and (A2) by using the Leibniz rule twice, we have

$$\vdash ((P \equiv Q)\&(R \equiv S)) \Rightarrow ((P \wedge R) \equiv (Q \wedge R)\&(Q \wedge R) \equiv (Q \wedge S)).$$

Hence, by Lemma 5.2(j) and Lemma 5.2(h), we obtain

$$\vdash ((P \equiv Q)\&(R \equiv S)) \Rightarrow ((P \wedge R) \equiv (Q \wedge S).$$

(q)

$((Q \vee R) \equiv (Q \vee R))\&(P \equiv Q) \Rightarrow ((Q \vee R) \equiv (R \vee P)) \qquad (\text{A}_\text{s}4)$

$\Leftrightarrow \langle(\text{Leib}) + \text{Lemma 5.2}((\text{a}) + (\text{b})) + \text{Rule (T)}; \text{"C} - \text{part"}:$

$\quad \mathbf{p}\&(P \equiv Q) \Rightarrow ((Q \vee R) \equiv (R \vee P))\rangle$

$\top\&(P \equiv Q) \Rightarrow ((Q \vee R) \equiv (R \vee P))$

$\Leftrightarrow \langle(\text{Leib}) + (\text{A}_\text{s}5); \text{"C} - \text{part"}: \mathbf{p} \Rightarrow ((Q \vee R) \equiv (R \vee P))\rangle$

$(P \equiv Q) \Rightarrow ((Q \vee R) \equiv (R \vee P))$

$\Leftrightarrow \langle(\text{Leib}) + \text{Lemma 5.2}(\text{d}); \text{"C} - \text{part"}: (P \equiv Q) \Rightarrow \mathbf{p}\rangle$

$(P \equiv Q) \Rightarrow ((R \vee P) \equiv (Q \vee R))$

$\Leftrightarrow \langle(\text{Leib}) + \text{item (a)}; \text{"C} - \text{part"}: (P \equiv Q) \Rightarrow (\mathbf{p} \equiv (Q \vee R))\rangle$

$(P \equiv Q) \Rightarrow ((P \vee R) \equiv (Q \vee R))$ ■

Remark 6.3.

Items (c), (e) and (f) in Lemma 5.3 have been proved in the basic EQ-logic and we prove them again without need to goodness axiom (A1).

Lemma 5.4.

(a) $\vdash (P \equiv Q) \equiv ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$;

(b) $\vdash (P \& Q) \Rightarrow (P \equiv Q)$;

(c) $\vdash \mathbf{\Delta}(P \equiv Q) \Rightarrow (\mathbf{\Delta}P \equiv \mathbf{\Delta}Q)$;

(d) $\vdash (P \Rightarrow S) \Rightarrow ((S \Rightarrow Q) \Rightarrow (P \Rightarrow Q))$;

(e) $\vdash ((P \vee Q) \Rightarrow R) \equiv ((P \Rightarrow R) \wedge (Q \Rightarrow R))$;

(f) $(P \Rightarrow R), (Q \Rightarrow R) \vdash ((P \vee Q) \Rightarrow R)$;

(g) $\vdash ((P \wedge Q) \Rightarrow R) \equiv ((P \Rightarrow R) \vee (Q \Rightarrow R))$;

(h) $\vdash \mathbf{\Delta}(P \wedge Q) \equiv (\mathbf{\Delta}P \wedge \mathbf{\Delta}Q)$;

(i) $\vdash \mathbf{\Delta}(P \vee Q) \equiv (\mathbf{\Delta}P \vee \mathbf{\Delta}Q)$;

(j) $\vdash (P \& \mathbf{\Delta}(P \Rightarrow Q)) \Rightarrow Q$ and $\vdash (\mathbf{\Delta}(P \Rightarrow Q) \& P) \Rightarrow Q$;

(k) $\vdash (P \& \mathbf{\Delta}(P \equiv Q)) \Rightarrow Q$ and $\vdash (\mathbf{\Delta}(P \equiv Q) \& P) \Rightarrow Q$;

(l) $\vdash \mathbf{\Delta}(P \equiv Q) \Rightarrow ((P \& R) \equiv (Q \& R))$ and
$\vdash \mathbf{\Delta}(P \equiv Q) \Rightarrow ((R \& P) \equiv (R \& Q))$.

(m) $\vdash \mathbf{\Delta}(P \Rightarrow Q) \Rightarrow ((P \& R) \Rightarrow (Q \& R))$ and
$\vdash \mathbf{\Delta}(P \Rightarrow Q) \Rightarrow ((R \& P) \Rightarrow (R \& Q))$;

(n) $\vdash \mathbf{\Delta}Q \Rightarrow (R \Rightarrow (Q \& R))$ and $\vdash \mathbf{\Delta}Q \Rightarrow (R \Rightarrow (R \& Q))$.

Proof.

(a) From Lemma 5.2(g), (f) and (h), it is easy to see that

$$\vdash (P \Rightarrow Q) \Rightarrow ((Q \Rightarrow P) \Rightarrow (P \equiv Q)).$$

By this, Lemma 5.2(n) and Lemma 5.3(f) using Lemma 5.2(h), we get

$$\vdash (P \Rightarrow Q) \Rightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P) \Rightarrow (P \equiv Q)).$$

Similarly, $\vdash (Q \Rightarrow P) \Rightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P) \Rightarrow (P \equiv Q))$. Then, by Conclusion Lemma 5.3(k), we obtain

$$\vdash ((P \Rightarrow Q) \wedge (Q \Rightarrow P) \Rightarrow (P \equiv Q)).$$

Hence, by this, Lemma 5.2(m), Lemma 5.2(c) and Lemma 5.3(c) by (**MP**) the result holds.

(b) Using Lemma 5.2(h) with Lemma 5.2(k) and Lemma 5.3(m), we get

$$\vdash (P \& Q) \Rightarrow (P \Rightarrow Q).$$

Similarly, $\vdash (P \& Q) \Rightarrow (Q \Rightarrow P)$. From this, and Lemma 5.2(l), we obtain

$$\vdash (P \& Q) \Rightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P)).$$

Then the Leibniz rule with item (a) yields the result.

(c) By Lemma 5.2(g), Necessitation rule (**N**) and (A_s10) by (**MP**), we obtain

$$\vdash \boldsymbol{\Delta}(P \equiv Q) \Rightarrow \boldsymbol{\Delta}(P \Rightarrow Q) \text{ and } \vdash \boldsymbol{\Delta}(P \equiv Q) \Rightarrow \boldsymbol{\Delta}(Q \Rightarrow P).$$

Then by Lemma 5.2(h) with (A_s10), we get

$$\vdash \boldsymbol{\Delta}(P \equiv Q) \Rightarrow (\boldsymbol{\Delta}P \Rightarrow \boldsymbol{\Delta}Q) \text{ and } \vdash \boldsymbol{\Delta}(P \equiv Q) \Rightarrow (\boldsymbol{\Delta}Q \Rightarrow \boldsymbol{\Delta}P).$$

From this and Lemma 5.2(l), we obtain

$$\vdash \boldsymbol{\Delta}(P \equiv Q) \Rightarrow ((\boldsymbol{\Delta}P \Rightarrow \boldsymbol{\Delta}Q) \wedge (\boldsymbol{\Delta}Q \Rightarrow \boldsymbol{\Delta}P)).$$

From this and item (a), by (Leib) rule, we obtain the result.

(d) From Lemma 5.2(f) in the form

$$\vdash (P \equiv (P \wedge S)) \Rightarrow ((P \equiv (P \wedge S) \wedge Q) \equiv ((P \wedge S) \equiv (P \wedge S) \wedge Q)),$$

and associativity and commutativity of "$\wedge$" and Lemma 5.2(d) by the Leibniz rule, we get

$$\vdash (P \Rightarrow S) \Rightarrow ((P \wedge S) \Rightarrow Q) \equiv (P \Rightarrow (Q \wedge S))).$$

From this and Lemma 5.2(g) by Lemma 5.2(h), we obtain

$$\vdash (P \Rightarrow S) \Rightarrow ((P \wedge S) \Rightarrow Q) \Rightarrow (P \Rightarrow (Q \wedge S))).$$

From this, (A11) and double Lemma 5.3(e) using Lemma 5.2(h), we get

$$\vdash (P \Rightarrow S) \Rightarrow (((P \wedge S) \Rightarrow Q) \Rightarrow (P \Rightarrow Q)).$$

On the other hand, by Lemma 5.2(e) and Lemma 5.3(f), we obtain

$$\vdash (((P \wedge S) \Rightarrow Q) \Rightarrow (P \Rightarrow Q)) \Rightarrow ((S \Rightarrow Q) \Rightarrow (P \Rightarrow Q)).$$

Hence, by Lemma 5.2(h) the result holds.

(e) From item (d), we have

$$\vdash ((P \vee Q) \Rightarrow Q) \Rightarrow ((Q \Rightarrow R) \Rightarrow ((P \vee Q) \Rightarrow R))$$

By this, Lemma 5.2(n) and Lemma 5.3(f) using Lemma 5.2(h), we get

$$\vdash ((P \vee Q) \Rightarrow Q) \Rightarrow ((P \Rightarrow R) \wedge (Q \Rightarrow R) \Rightarrow ((P \vee Q) \Rightarrow R)).$$

From this, Lemma 5.3(d) and Lemma 5.2(a) using (Leib), we obtain

$$\vdash (P \Rightarrow Q) \Rightarrow ((P \Rightarrow R) \wedge (Q \Rightarrow R) \Rightarrow ((P \vee Q) \Rightarrow R)).$$

Similarly, $\vdash (Q \Rightarrow P) \Rightarrow ((P \Rightarrow R) \wedge (Q \Rightarrow R) \Rightarrow ((P \vee Q) \Rightarrow R))$. Then, by Conclusion Lemma 5.3(k), we obtain

$$\vdash (P \Rightarrow R) \wedge (Q \Rightarrow R) \Rightarrow ((P \vee Q) \Rightarrow R).$$

On the other hand, $\vdash ((P \vee Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \wedge (Q \Rightarrow R))$ easily follows from Lemma 5.3(b) and Lemma 5.3(f) by Lemma 5.2(l). Hence, by (**MP**) with Lemma 5.2(c) and Lemma 5.3(c), we get the result.

(f) Direct from the assumptions, Lemma 5.2(c), item (e) and Lemma 5.2(a) using (**EA**).

(g) From Lemma 5.3(b) and Lemma 5.3(e), we get

$$\vdash (((P \wedge Q) \Rightarrow R) \Rightarrow (Q \Rightarrow R)) \Rightarrow (((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R))).$$

From this, and item (d) in the form

$$\vdash (P \Rightarrow (P \wedge Q)) \Rightarrow (((P \wedge Q) \Rightarrow R) \Rightarrow ((Q \Rightarrow R))$$

by Lemma 5.2(h), we obtain

$$\vdash (P \Rightarrow (P \wedge Q)) \Rightarrow (((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R))).$$

By this, Lemma 5.3(l) and Lemma 5.2(a) by (Leib), we obtain

$$\vdash (P \Rightarrow Q) \Rightarrow (((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R))).$$

Similarly, $\vdash (Q \Rightarrow P) \Rightarrow (((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R)))$. Then, by Conclusion Lemma 5.3(k), we get

$$\vdash ((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R)).$$

On the other hand, from Lemma 5.2(n) using Lemma 5.3(f), we have

$$\vdash (P \Rightarrow R) \Rightarrow ((P \wedge Q) \Rightarrow R) \text{ and } \vdash (Q \Rightarrow R) \Rightarrow ((P \wedge Q) \Rightarrow R).$$

From this and item (f), we obtain

$$\vdash ((P \Rightarrow R) \vee (Q \Rightarrow R)) \Rightarrow ((P \wedge Q) \Rightarrow R).$$

Hence, by (**MP**) with Lemma 5.2(c) and Lemma 5.3(c), we get the result.

(h) From (A_s10) and Lemma 5.3(h), we get

$\vdash (\mathbf{\Delta}(P \Rightarrow (P \wedge Q)) \vee \mathbf{\Delta}(Q \Rightarrow (P \wedge Q))) \Rightarrow ((\mathbf{\Delta}P \Rightarrow \mathbf{\Delta}(P \wedge Q)) \vee (\mathbf{\Delta}Q \Rightarrow \mathbf{\Delta}(P \wedge Q)))$.

By this and Lemma 5.3(l) by (Leib) twice, we obtain

$\vdash (\mathbf{\Delta}(P \Rightarrow Q) \vee \mathbf{\Delta}(Q \Rightarrow P)) \Rightarrow ((\mathbf{\Delta}P \Rightarrow \mathbf{\Delta}(P \wedge Q)) \vee (\mathbf{\Delta}Q \Rightarrow \mathbf{\Delta}(P \wedge Q)))$.

From this by the Leibniz rule with item (g), we get

$\vdash (\mathbf{\Delta}(P \Rightarrow Q) \vee \mathbf{\Delta}(Q \Rightarrow P)) \Rightarrow ((\mathbf{\Delta}P \wedge \mathbf{\Delta}Q) \Rightarrow \mathbf{\Delta}(P \wedge Q))$.

Then, by (**MP**) with (A_s11), we obtain $\vdash (\mathbf{\Delta}P \wedge \mathbf{\Delta}Q) \Rightarrow \mathbf{\Delta}(P \wedge Q)$.

On the other hand, From Lemma 5.2(n) and (A_s10) using (**MP**), we obtain

$\vdash \mathbf{\Delta}(P \wedge Q) \Rightarrow \mathbf{\Delta}P$ and $\vdash \mathbf{\Delta}(P \wedge Q) \Rightarrow \mathbf{\Delta}Q$.

Then, by Lemma 5.2(l), we obtain $\vdash \mathbf{\Delta}(P \wedge Q) \Rightarrow (\mathbf{\Delta}P \wedge \mathbf{\Delta}Q)$. Hence, by (**MP**) with Lemma 5.2(c) and Lemma 5.3(c), we get the result.

(i) From Lemma 5.3(b), we get $\vdash \mathbf{\Delta}Q \Rightarrow (\mathbf{\Delta}P \vee \mathbf{\Delta}Q)$ and then Lemma 5.3(e), we obtain

$\vdash (\mathbf{\Delta}(P \vee Q) \Rightarrow \mathbf{\Delta}Q) \Rightarrow (\mathbf{\Delta}(P \vee Q) \Rightarrow (\mathbf{\Delta}P \vee \mathbf{\Delta}Q))$.

From this, (A_s10) and Lemma 5.3(f) by (**MP**), we get

$\vdash \mathbf{\Delta}((P \vee Q) \Rightarrow Q) \Rightarrow (\mathbf{\Delta}(P \vee Q) \Rightarrow (\mathbf{\Delta}P \vee \mathbf{\Delta}Q))$.

Hence, by (Leib) with Lemma 5.3(d) and Lemma 5.3(a), we obtain

$\vdash \mathbf{\Delta}(P \Rightarrow Q) \Rightarrow (\mathbf{\Delta}(P \vee Q) \Rightarrow (\mathbf{\Delta}P \vee \mathbf{\Delta}Q))$.

Similarly, $\vdash \mathbf{\Delta}(Q \Rightarrow P) \Rightarrow (\mathbf{\Delta}(P \vee Q) \Rightarrow (\mathbf{\Delta}P \vee \mathbf{\Delta}Q))$. Then, by Conclusion Lemma 5.3(i), we get $\vdash (\mathbf{\Delta}(P \vee Q) \Rightarrow (\mathbf{\Delta}P \vee \mathbf{\Delta}Q))$.

On the other hand, from Lemma 5.3(b), Necessitation (**N**) and (A_s10) using (**MP**), we get

$$\vdash \mathbf{\Delta} P \Rightarrow \mathbf{\Delta}(P \vee Q) \text{ and } \vdash \mathbf{\Delta} Q \Rightarrow \mathbf{\Delta}(P \vee Q).$$

Then, by item (f), we obtain $\vdash (\mathbf{\Delta} P \vee \mathbf{\Delta} Q) \Rightarrow \mathbf{\Delta}(P \vee Q)$. Hence, by (**MP**) with Lemma 5.2(c) and Lemma 5.3(c), we get the result.

(j) From Lemma 5.2(k) and Lemma 5.3(f), we get

$$\vdash (P \Rightarrow Q) \Rightarrow ((P \mathbin{\&} \mathbf{\Delta}(P \Rightarrow Q)) \Rightarrow Q).$$

From this and (A_s8) ($\vdash \mathbf{\Delta}(P \Rightarrow Q) \Rightarrow (P \Rightarrow Q)$) using Lemma 5.2(h), we obtain

$$\vdash \mathbf{\Delta}(P \Rightarrow Q) \Rightarrow ((P \mathbin{\&} \mathbf{\Delta}(P \Rightarrow Q)) \Rightarrow Q).$$

On the other hand, from (A_s6) and Lemma 5.3(e), we get

$$\vdash \neg\mathbf{\Delta}(P \Rightarrow Q) \Rightarrow (\mathbf{\Delta}(P \Rightarrow Q)) \Rightarrow Q),$$

and from Lemma 5.2(k) and Lemma 5.3(f), we obtain

$$\vdash (\mathbf{\Delta}(P \Rightarrow Q) \Rightarrow Q) \Rightarrow ((P \mathbin{\&} \mathbf{\Delta}(P \Rightarrow Q)) \Rightarrow Q).$$

Hence, by Lemma 5.2(h) we obtain

$$\vdash \neg\mathbf{\Delta}(P \Rightarrow Q) \Rightarrow ((P \mathbin{\&} \mathbf{\Delta}(P \Rightarrow Q)) \Rightarrow Q).$$

Then, by item (f) and (**MP**) with (A_s13), we get $\vdash (P \mathbin{\&} \mathbf{\Delta}(P \Rightarrow Q)) \Rightarrow Q$.

Similarly, $\vdash (\mathbf{\Delta}(P \Rightarrow Q) \mathbin{\&} P) \Rightarrow Q$.

(k) Direct from Lemma 5.2(g), Necessitation (**N**) and (A_s10) using (**MP**), we get

$$\vdash \mathbf{\Delta}(P \equiv Q) \Rightarrow \mathbf{\Delta}(P \Rightarrow Q)$$

By this, (A4) and Lemma 5.2(i), we obtain

$$\vdash P\&\boldsymbol{\Delta}(P \equiv Q) \Rightarrow P\&\boldsymbol{\Delta}(P \Rightarrow Q) \text{ and } \vdash \boldsymbol{\Delta}(P \equiv Q)\&P \Rightarrow \boldsymbol{\Delta}(P \Rightarrow Q)\&P$$

Hence, from item (j) using Lemma 5.2(h) the result holds.

(l)

$\boldsymbol{\Delta}(P \equiv Q) \Rightarrow ((\top\&P)\&R) \equiv (\top\&(Q\&R))$ $\quad$ (A_s13)

$\Leftrightarrow \langle(\text{Leib})\text{twice} + (\text{A5})\rangle$

$\boldsymbol{\Delta}(P \equiv Q) \Rightarrow ((P\&R) \equiv (Q\&R))$

The second part follows exactly by the similar proof as above.

(m) By item (l), we get

$$\vdash \boldsymbol{\Delta}((P \wedge Q) \equiv P) \Rightarrow (((P \wedge Q)\&R) \equiv (P\&R)).$$

Then, by Lemma 5.2(g) and (h), we obtain

$$\vdash \boldsymbol{\Delta}((P \wedge Q) \equiv P) \Rightarrow ((P\&R) \Rightarrow ((P \wedge Q)\&R)).$$

By this, (A7), double Lemma 5.3(e) using (**MP**), we have

$$\vdash \boldsymbol{\Delta}(P \Rightarrow Q) \Rightarrow ((P\&R) \Rightarrow ((Q\&R)).$$

Similarly, $\vdash \boldsymbol{\Delta}(P \Rightarrow Q) \Rightarrow ((R\&P) \Rightarrow (R\&Q))$.

(n) From (A_s5), and (A_s10) using (**N**) and then (**MP**), we get

$$\vdash \boldsymbol{\Delta}Q \Rightarrow \boldsymbol{\Delta}(\top \equiv Q).$$

From this, and (A6) by (Leib), we obtain

$$\vdash \boldsymbol{\Delta}Q \Rightarrow \boldsymbol{\Delta}(\top \Rightarrow Q).$$

Hence, by Lemma 5.2(h) with item (m), we obtain

$$\vdash \boldsymbol{\Delta}Q \Rightarrow ((\top\&R) \Rightarrow (Q\&R)).$$

Thus, by the Leibniz rule with (A5), we get

$$\vdash \mathbf{\Delta}Q \Rightarrow (R \Rightarrow (Q\mathbf{\&}R)).$$

Similarly, $\vdash \mathbf{\Delta}Q \Rightarrow (R \Rightarrow (R\mathbf{\&}Q))$. ■

We extend to $\ell\mathrm{EQ}^{\mathrm{s}}_{\mathbf{\Delta}}$-logic the following result. The proof is completely the same as in [5]. We shall supply the proof because of the importance of the statement and to make the paper self-contained:

Lemma 5.5.

(a) $\vdash (\mathbf{\Delta}P\mathbf{\&}\mathbf{\Delta}P) \equiv \mathbf{\Delta}P$;
(b) $\vdash \mathbf{\Delta}(P \equiv Q)\mathbf{\&}\mathbf{\Delta}(R \equiv S) \Rightarrow ((P\mathbf{\&}R) \equiv (Q\mathbf{\&}S))$.

Proof.

(a) By Lemma 5.4(n), we get

$$\vdash \mathbf{\Delta}P \Rightarrow (\mathbf{\Delta}P \Rightarrow (\mathbf{\Delta}P\mathbf{\&}\mathbf{\Delta}P)).$$

On the other hand, by ($\mathrm{A_s}$6) and Lemma 5.3(e), we obtain

$$\vdash \neg\, \mathbf{\Delta}P \Rightarrow (\mathbf{\Delta}P \Rightarrow (\mathbf{\Delta}P\mathbf{\&}\mathbf{\Delta}P)).$$

Hence, Lemma 5.4(f) and ($\mathrm{A_s}$13) by (**MP**) the result holds.

(b) Direct from Lemma 5.4(l) and Lemma 5.2(i) by the transitivity of " $\equiv$ ".■

5.2 $\ell\mathrm{EQ}^{\mathrm{s}}_{\mathbf{\Delta}}$-logic: semantics

Definition 5.2.

Interpretation of $\ell\mathrm{EQ}^{\mathrm{s}}_{\Delta}$-logic is a tuple $\mathfrak{R} = (\mathcal{E}_\Delta, e)$ in which $\mathcal{E}_\Delta = (E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ is $\ell\mathrm{EQ}^{\mathrm{s}}_{\Delta}$-algebra and a function $e: F_{\mathcal{T}} \to E$ called the *truth evaluation* of the interpretation that satisfies the following identities for all formulas $P, Q \in F_{\mathcal{T}}$:

$$e(\top) = \mathbf{1}; \quad e(\bot) = \mathbf{0};$$
$$e(P \boldsymbol{\wedge} Q) = e(P) \wedge e(Q);$$
$$e(P \boldsymbol{\vee} Q) = e(P) \vee e(Q);$$
$$e(P \boldsymbol{\&} Q) = e(P) \otimes e(Q);$$
$$e(P \equiv Q) = e(P) \sim e(Q);$$
$$e(\boldsymbol{\Delta} P) = \Delta e(P).$$

Let T be a theory and $\mathfrak{R} = (\mathcal{E}_\Delta, e)$ be an interpretation, then

If $\mathfrak{R} \vDash P$ for all $P \in T$, we write $\mathfrak{R} \vDash T$,

and we say that $\mathfrak{R}$ is a $\mathcal{E}_\Delta$-*model* of T.

Lemma 5.6.

The inference rules of $\ell\mathrm{EQ}^{\mathrm{s}}_{\boldsymbol{\Delta}}$-logic are sound in the following sense. Let a tuple $\mathfrak{R} = (\mathcal{E}_\Delta, e)$ in which $\mathcal{E}_\Delta$ is $\ell\mathrm{EQ}^{\mathrm{s}}_{\Delta}$-algebra and a function $e: F_{\mathcal{T}} \to E$ called the *truth evaluation* of the interpretation:

(a) If $e(P \equiv Q) = \mathbf{1}$ then, $e(C[\mathbf{p}:= Q] \equiv C[\mathbf{p}:= R]) = \mathbf{1}$ for any formula P;
(b) If $e(P) = \mathbf{1}$ and $e(P \Rightarrow Q) = \mathbf{1},$ then $e(Q) = \mathbf{1}$.

Proof.

It has been proved that Leibniz is sound in the setting of basic EQ-logic [6] (see Lemma 3.9).

(b) Suppose that $e(P) = \mathbf{1}$ and $e(P \Rightarrow Q) = \mathbf{1},$ then

$$\begin{aligned} e(P \Rightarrow Q) &= e(P) \sim (e(P) \wedge e(Q)) \\ &= e(P) \to e(Q) = \mathbf{1} \to e(Q) \\ &= \mathbf{1} \sim (\mathbf{1} \wedge e(Q)) = \mathbf{1} \sim e(Q) = \mathbf{1} \end{aligned}$$

Then, necessarily $e(Q) = \mathbf{1}$. ∎

It is straightforward using Lemma 3.8, the axioms and the properties of $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras to see all the logical axioms of the $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-logic are tautologies.

The following is standard procedure due to Lindenbaum and Tarski, we now address the completeness of the $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-logic.

Let T be a theory over the $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-logic. Put

$$P \approx Q \ \text{ iff } \ T \vdash P \equiv Q, \ P, Q \in F_{\mathcal{T}}$$

It follows from Lemma 5.2(b), Lemma 5.2(d) and Lemma 5.2(j) that " $\approx$ " is an equivalence relation on $F_{\mathcal{T}}$.

Let $\rho: F_{\mathcal{T}} \to F_{\mathcal{T}}/\approx$ be the quotient map onto the set of all equivalence classes $|P| = \{Q \mid T \vdash P \equiv Q\}$. The Leibniz rule (Leib) guarantees that the logical connectives possess the substitution property for " $\approx$ ". In consequence, the following operations are well defined on the set $\bar{E} = \{|P| | P \in F_{\mathcal{T}}\}$:

$|P| \wedge_T |Q| = \rho(P \wedge Q)$;
$|P| \vee_T |Q| = \rho(P \vee Q)$;
$|P| \otimes_T |Q| = \rho(P \& Q)$;
$|P| \sim_T |Q| = \rho(P \equiv Q)$;
$\Delta_T |Q| = \rho(\boldsymbol{\Delta} P)$.

The partial order $\leq$ is also well-defined on $F_{\mathcal{T}}/\approx$ by

$$|P| \leq |Q| \ \text{ iff } \ |P| \wedge_T |Q| = |P| \text{ iff } \ T \vdash P \wedge Q \equiv P \ \text{ iff } \ T \vdash P \Rightarrow Q.$$

Let $\mathcal{E}_T = \langle \bar{E}, \wedge_T, \vee_T, \otimes_T, \sim_T, \Delta_T, \mathbf{0}_T, \mathbf{1}_T \rangle$ be the Lindenbaum algebra of the theory T, where $\mathbf{1}_T = \rho(\top), \mathbf{0}_T = \rho(\bot)$. By virtue of

Lemma 5.1-Lemma 5.4, $\mathcal{E}_T$ is $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebra and the top element $\mathbf{1}_T$ is exactly the equivalence class $\{P \in F_{\mathcal{T}} \mid T \vdash P\}$. Moreover, the quotient map is a truth evaluation and the separateness holds as follows:

Let $|P| \sim_T |Q| = \mathbf{1}$, then $\mathbf{1} = |P| \sim_T |Q| = \rho(P \equiv Q) = \rho(P) \sim \rho(Q)$. Then, necessarily $\rho(P) = \rho(Q)$; that is $|P| = |Q|$.

From these arguments with the representation theorem (Theorem 4.7), we deduce the following theorem.

Theorem 5.1. (**Completeness**)

The prelinear $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-logic is generally complete and chain complete for the variety of prelinear $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras. Specifically, for every formula $P \in F_J$ and for every theory T over the prelinear $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-logic the following are equivalent:

(a) $T \vdash P$.

(b) For each prelinear $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebra $\mathcal{E}_\Delta$ and each $\mathcal{E}_\Delta$-model of $\mathfrak{R}$ of T, $\mathfrak{R} \vDash$ A.

(c) For each linearly ordered $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebra $\mathcal{E}_\Delta$ and each $\mathcal{E}_\Delta$-mode $\mathfrak{R}$ of T, $\mathfrak{R} \vDash$ A.

Theorem 5.2. (Deduction theorem)

For each theory T, formula P and arbitrary formula Q it holds that

$$T \cup \{P\} \vdash Q \quad \text{iff} \quad T \vdash \boldsymbol{\Delta} P \Rightarrow Q$$

Proof.

Let $T \cup \{P\} \vdash Q$. The proof follows by induction on the proof length of Q.

(a) If $Q := P$, $Q \in T$ or Q is a logical axiom, then ($\mathrm{A_s}8$) and Lemma 5.3(m) lead to the result.

(b) Let Q have been obtained using the rule (**EA**) by the proof

$$\ldots, R,\ R \equiv Q,\ Q.$$

Then, from the inductive hypothesises

$$T \vdash \boldsymbol{\Delta} P \Rightarrow R, \text{ and } T \vdash \boldsymbol{\Delta} P \Rightarrow (R \equiv Q),$$

the Necessitation rule (**N**) and (A_s10) using (**MP**), we have

$$T \vdash \mathbf{\Delta}P \Rightarrow R, \text{ and } T \vdash \mathbf{\Delta\Delta}P \Rightarrow \mathbf{\Delta}(R \equiv Q).$$

From this and (Leib) with (A_s9), we obtain

$$T \vdash \mathbf{\Delta}P \Rightarrow R, \text{ and } T \vdash \mathbf{\Delta}P \Rightarrow \mathbf{\Delta}(R \equiv Q).$$

From this and Lemma 5.2(i), we get

$$T \vdash (\mathbf{\Delta}P \mathbf{\&} \mathbf{\Delta}P) \Rightarrow (R \mathbf{\&} \mathbf{\Delta}(R \equiv Q)).$$

By this, Lemma 5.5(a) and Lemma 5.4(k) using Lemma 5.2(h), we have $T \vdash \mathbf{\Delta}P \Rightarrow Q$.

(c) Let $Q := S[\mathbf{p} := U] \equiv S[\mathbf{p} := V]$ have been obtained using the Leibniz rule (Leib) by the proof

$$\ldots, U \equiv V,\ S[\mathbf{p} := U] \equiv S[\mathbf{p} := V].$$

Then, the proof proceeds by induction on the complexity of the formula S:

(i) If S is $\bot$, then

$$S[\mathbf{p} := U] \equiv S[\mathbf{p} := V] \text{ is } S \equiv S.$$

Using (**MP**) with Lemma 5.3(m):

$$T \vdash (S \equiv S) \Rightarrow (\mathbf{\Delta}P \Rightarrow (S \equiv S)),$$

we have $T \vdash \mathbf{\Delta}P \Rightarrow (S \equiv S)$.

(ii) If S is $\mathbf{p}$ then it follows directly from the inductive hypothesis.

(iii) Let S be $G \mathbin{\Box} H$, where $\Box \in \{\wedge, \vee, \mathbf{\&}, \equiv\}$. Then we must prove that

$$T \vdash \mathbf{\Delta}P \Rightarrow ((G \mathbin{\Box} H)[\mathbf{p} := U] \equiv (G \mathbin{\Box} H)[\mathbf{p} := V]).$$

That is

$$T \vdash \mathbf{\Delta}P \Rightarrow ((G' \mathbin{\Box} H') \equiv (G'' \mathbin{\Box} H'')) \tag{5.2}$$

where

$$G' := G[\mathbf{p} := E],\ H' := H[\mathbf{p} := E]$$
$$G'' := G[\mathbf{p} := F],\ H'' := H[\mathbf{p} := F].$$

By the inductive assumptions,

$$T \vdash \mathbf{\Delta}P \Rightarrow (G' \equiv G'') \text{ and } T \vdash \mathbf{\Delta}P \Rightarrow (H' \equiv H'').$$

Thus, in case that $\square \in \{\wedge, \equiv\}$, from Lemma 5.2(i), we have

$$T \vdash (\mathbf{\Delta}P \& \mathbf{\Delta}P) \Rightarrow (G' \equiv G'')\&(H' \equiv H'').$$

By this, and (Leib) with Lemma 5.5(a), we get

$$T \vdash \mathbf{\Delta}P \Rightarrow (G' \equiv G'')\&(H' \equiv H'').$$

Thus, (5.2) follows by Lemma 5.2(h) with Lemma 5.2(o)

$$T \vdash \mathbf{\Delta}P \Rightarrow (G' \equiv H') \equiv (G'' \equiv H'').$$

Similarly, using Lemma 5.3(p) $T \vdash \mathbf{\Delta}P \Rightarrow (G' \wedge H') \equiv (H'' \wedge G'')$.

In case that $\square$ is "**&**", from rule (**N**), (**MP**) with (A_s10), we get

$$T \vdash \mathbf{\Delta\Delta}P \Rightarrow \mathbf{\Delta}(G' \equiv G'') \text{ and } T \vdash \mathbf{\Delta\Delta}P \Rightarrow \mathbf{\Delta}(H' \equiv H'').$$

By this, and (Leib) with (A_s9), we obtain

$$T \vdash \mathbf{\Delta}P \Rightarrow \mathbf{\Delta}(G' \equiv G'') \text{ and } T \vdash \mathbf{\Delta}P \Rightarrow \mathbf{\Delta}(H' \equiv H'').$$

Hence, from Lemma 5.2(i), and the Leibniz (Leib) with Lemma 5.5(a), we have

$$T \vdash \mathbf{\Delta}P \Rightarrow \mathbf{\Delta}(G' \equiv G'')\&\mathbf{\Delta}(H' \equiv H'')$$

Thus, (5.2) follows from Lemma 5.5(b) using Lemma 5.2(h). In case that $\square$ is $\vee$, from Lemma 5.3(q) and Lemma 5.2(i), we get

$T \vdash ((G' \equiv G'')\&(H' \equiv H'')) \Rightarrow$

$$(((G'\vee H') \equiv (G''\vee H'))\&((G''\vee H') \equiv (G''\vee H'')))$$

By this and the transitivity of " $\equiv$ " using Lemma 5.2(h), we have

$$T \vdash ((G' \equiv G'')\&(H' \equiv H'')) \Rightarrow ((G'\vee H') \equiv (G''\vee H'')).$$

Hence, by this, the inductive assumptions, Lemma 5.2(i) and Lemma 5.5(a) using Lemma 5.2(h), (5.2) holds.

(iv) Let S be $\boldsymbol{\Delta}H$. Then we have

(L.1) $T \vdash \boldsymbol{\Delta}P \Rightarrow (H' \equiv H'')$ (Inductive assumption)

(L.2) $T \vdash \boldsymbol{\Delta\Delta}P \Rightarrow \boldsymbol{\Delta}(H' \equiv H'')$ ((L.1), rule (**N**), (A_s10) and (**MP**))

(L.3) $T \vdash \boldsymbol{\Delta}P \Rightarrow \boldsymbol{\Delta}(H' \equiv H'')$ ((L.2), (Leib), and (A_s9))

(L.4) $T \vdash \boldsymbol{\Delta}P \Rightarrow (\boldsymbol{\Delta}H' \equiv \boldsymbol{\Delta}H'')$ ((L.3), Lemma 5.4(c), Lemma 5.2(h))

(d) Let $Q := \boldsymbol{\Delta}R$ have been obtained using rule (**N**) by the proof

$$\ldots, R, \boldsymbol{\Delta}R.$$

Then, from the inductive assumptions: $T \vdash \boldsymbol{\Delta}P \Rightarrow R$, the Necessitation rule (**N**), and (A_s10) using (**MP**), we get: $T \vdash \boldsymbol{\Delta\Delta}P \Rightarrow \boldsymbol{\Delta}R$. From this and (Leib) with (A_s9), we obtain $T \vdash \boldsymbol{\Delta}P \Rightarrow \boldsymbol{\Delta}R$. Hence, by Lemma 5.2(h) with (A_s8), we get the result.

The converse implication is obtained using rules (**N**) and (**MP**). ■

Remark 5.3:

One of the useful properties of $\boldsymbol{\Delta}$-connective is that the deduction theorem cannot be proved without introducing it. It is also necessary to develop the predicate $\ell EQ_{\boldsymbol{\Delta}}^{s}$-logic.

Chapter 6
Conclusion and Future Work

We continue in this thesis the study of EQ-algebras, begun in [7, 8, 22, 23]. We introduced and studied a class of separated (not necessarily good) lattice EQ-algebras that may be represented as subalgebras of products of linearly ordered ones. Such algebras are called representable. Namely, we enriched separated lattice EQ-algebras with a unary operation (the so called Baaz delta), fulfilling some additional assumptions. The resulting algebras are called ℓEQ_{Δ}^{s}-algebras. One of the main results of this thesis is to characterize the class of representable ℓEQ_{Δ}^{s}-algebras. We showed that prelinearity alone characterizes the representable class of ℓEQ_{Δ}^{s}-algebras. We also supplied a number of useful results, leading to this characterization. We also formulated the corresponding $\ell EQ_{\boldsymbol{\Delta}}^{s}$-logic and established its completeness for the semantical domain of ℓEQ_{Δ}^{s}-algebras. We in detail introduced syntax and semantics of the $\ell EQ_{\boldsymbol{\Delta}}^{s}$-logic and prove various theorems characterizing its properties including deduction theorem.

Finally, let us remark that $\ell EQ_{\boldsymbol{\Delta}}^{s}$-logic open the door for developing predicate $\ell EQ_{\boldsymbol{\Delta}}^{s}$-logic; also to introduce and study a class of $\ell EQ_{\boldsymbol{\Delta}}^{s}$-logics whose semantical domain based on separated (need not to be good) EQ-algebras.

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الملخص العربى

مؤخراً تم استحداث نظرية شكلية لفئة جديدة من منطق متعدد القيم ويسمى بمنطق المساواة (EQ-logic)، ويعتمد نطاق معانيه الجبرية على جبر خاص يسمى بجبر المساواة (EQ-algebra) والذى قدم طريقة بديلة لتطوير منطق فازى يعتمد على التكافؤ بدلا من التضمين. ويمكن اعتبار هذا الاتجاه تعميم لمنطق المساواة التقليدى (Equational propositional logic) المبرر بالفكرة التى قدمها لايبنتز والتى تنص على أن " الحساب المنطقى الأنسب يجب أن يعتمد على المساواة ". بالاضافة الى أن البراهين فى هذا النمط تكون أكثر فاعلية ووضوحاً.

يستهدف هذا العمل مواصلة البحث فى منطق المساواة ونطاق معانيه الجبرية والذى يعتبر نوع خاص من المنطق الفازى حيث السمة الأساسية هى الاكتمال للتركيبات الجبرية المرتبة خطياً. تحديداً، يتم دراسة و توصيف جبريات المساواة المنفصلة ذات الترتيب الشبكي (Separated lattice EQ-algebras) والقابلة للتمثيل كجبر جزئى من حاصل الضرب المباشر لجبرياته المرتبة خطياً (Representable) وعرض لأهم النتائج التي تؤدي إلى هذا التوصيف. وهذا من شأنه أن يسمح لتطوير منطق مساواة فازى أعم ذو معانى جيرية منفصلة يعتمد على التكافؤ بدلا من التضمين. تحديداً، صياغة منطق مساواة منفصل سمته الأساسية هى الاكتمال لجبريات المساواة المنفصلة ذات الترتيب الشبكي المرتبة خطياً وعرض بالتفصيل للخصائص والنظريات المختلفة لهذا المنطق والتي من ضمنها خاصية الإكتمال.

وقد اشتملت هذه الرسالة على ستة فصول كالتالى:

الفصل الأول: يشمل هذا الفصل على مقدمة مختصرة عن موضوع الرسالة والدوافع وراء هذا البحث مع عرض لمحتويات الرسالة.

الفصل الثانى: يقدم هذا الفصل ملخص للبناء اللغوى (Syntax) ودلالات (Semantics) منطق المساواة التقليدي. وعلاوة على ذلك، تقديم جميع التعريفات والمفاهيم الأساسية للصيغة المنطقية (Formula)، والبديهيات المنطقية، وقواعد الاستدلال. في حين تم تقديم أيضا ملاحظات قصيرة على صدق واكتمال منطق المساواة التقليدي (Soundness and completeness).

الفصل الثالث: ينقسم هذا الفصل إلى جزأين؛ يتم تخصيص الجزء الأول بشكل رئيسي لتقديم دراسة استقصائية عن جبريات المساواة من تعريفات وأنواع وخصائص أساسية هامة، وكذلك بعض الأمثلة

على جبريات المساواة. وأخيرا، يتم تقديم توصيف جبريات المساواة في وجود الرابط دلتا "Δ" وبدونه. الجزء الثاني مخصص لتقديم نظرة عامة على منطق المساواة الأساسي وعرض خصائصه الأساسية والذي يعتمد نطاق معانية الجيرية على جبريات المساواة. أيضا، يتم تقديم نظرية اكتمال هذا المنطق.

الفصل الرابع: يقدم هذا الفصل نوع خاص من جبريات المساواة تسمى جبريات المساواة المنفصلة ذات الترتيب الشبكي (ℓEQ_{Δ}^{s}-algebras) في وجود الرابط دلتا. بالإضافة إلى دراسة متعمقة للمرشحات والتطابقات. وعلاوة على ذلك، عرض توصيف لهذه الجبريات والتي قابلة للتمثيل.

الفصل الخامس: يقدم هذا الفصل منطق المساواة المنفصل ($\ell EQ_{\boldsymbol{\Delta}}^{s}$-logic) نطاق معانيه الجبرية يعتمد علي جبريات المساواة المنفصلة ذات الترتيب الشبكي وإثبات خصائصه الأساسية بما في ذلك نظرية الاكتمال ونظرية الإستنباط (Deduction theorem).

الفصل السادس: يشمل هذا الفصل النتائج التي تم الحصول عليها فى هذه الرسالة والأعمال المستقبلية المقترحة.

وفي نهاية الرسالة يوجد قائمة بالمراجع.

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موضوعات فى منطق المساواة الفازى ونطاق معانيه الجبرية

رسالة مقدمة كجزء من متطلبات الحصول على درجة الماجستير فى العلوم الهندسية الأساسية في الرياضيات الهندسية

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