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Stress-Strength Reliability of Some Statistical Distributions

A Thesis Submitted in Partial Fulfillment of the Requirements for the
MSc Degree in Engineering Mathematics and Physics
(Engineering Mathematics)

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Dedication

I would like to dedicate my thesis to my **husband**, my **parents**, my sons (**Malek**and **Hamza**), my **brother**, and my **sisters** for their patience, sacrifices, and support to complete this research work.

Summary

Stress-strength model is one of the most important models that measures the reliability of the product in many practicing engineers. The problem of the stress-strength model is originated in the context of reliability of a component has strength X subject to a stress Y, the component failing if and only if at any time the applied stress is greater than its strength. In this case the stress-strength reliability function is noted by $R = P(Y < X)$.

In this thesis, we study the statistical inference for the stress-strength parameters $R = P(Y < X)$, for two different statistical distributions from complete samples. The estimation of R for Quasi Lindley and Exponentiated Generalized Inverse Weibull distributions is proposed. Two methods of estimation are suggested; Maximum Likelihood estimation (MLE), and Bayesin estimation methods. Also, the asymptotic confidence interval for R based on the MLE is obtained. Bayesian estimator of R is obtained using two methods of Markov Chain Monte Carlo (MCMC) technique; (Importance Sampling, Metroplis Hastings), under different loss functions. Simulation is used for the purpose of illustration and comparing the different estimators according to the bias and the mean square error (MSE), also some real data examples are presented.

The thesis consists of three chapters:

Chapter 1: this chapter represents an overview of the research work undertaken in this thesis. A simple definitions and concepts for reliability, stress-strength model, multi-component stress-strength reliability are introduced . Some methods of the parameter estimation are presented. We mention the Monte Carlo techniques, which will be used in the next chapters for developing Monte Carlo approximations for Bayesian estimation. We also provide some important distributions that used in work. At the end of this chapter a literature of the previous studies is presented.

Chapter 2: in this chapter, we discuss the estimation of the stress-strength reliability using the maximum likelihood and Bayesian estimation methods, when X and Y both follow a Quasi Lindley distribution (QLD) with different parameters. Multi-component stress-strength reliability

function is also derived. Stress-strength reliability is studied using the maximum likelihood, and Bayes estimations. We obtained the 95% asymptotic confidence intervals of R. Bayesian estimations were proposed using two different methods: Importance Sampling technique and Metropolis-Hastings algorithm, under symmetric loss function (squared error) and asymmetric loss functions (linex, general entropy). The behaviors of the maximum likelihood and Bayes estimators of stress-strength reliability have been studied through the Monte Carlo simulation study. Finally analysis of a real data set has also been presented.

The results of this chapter were published at:
"Journal of Advances in Systems Science and Applications (ASSA), 2018, 4, 39-51."

Chapter 3: this chapter presents the stress-strength reliability when X and Y have an Exponentiated Generalized Inverse Weibull distribution (EGIW) with different parameters. The problem of stress-strength reliability is studied to obtain the reliability function of the parameters of EGIW distribution. Reliability for multi-component stress-strength model for EGIW distribution is also studied. Maximum likelihood estimation for stress-strength reliability is performed. Bayesian estimator of R is obtained using Importance Sampling technique under the squared error loss function. A simulation study to investigate and compare the performance of each method of estimation is performed. Finally analysis of a real data set has also been presented for illustrative purposes.

The results of this chapter were published at:
"Journal of Statistics Applications and Probability, 2018, 7, 1-10."

The lists of references arranged alphabetically, and publications out of this research study are provided towards the end of the thesis.

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Notations

$F(.)$	Cumulative distribution function.
$E(.)$	Expected value.
$h(.)$	Hazard rate function.
$L(.)$	Likelihood function.
$l(.)$	Log-likelihood function.
$Ls(.)$	Loss function.
$R_{s,k}$	Multi-component stress-strength reliability.
$N(.,.)$	Normal distribution .
$f(.)$	Probability density function.
n	Sample size of X.
m	Sample size of Y.
$U(0,1)$	Standard uniform distribution.
R	Stress-strength reliability.
$S(.)$	Survival function.
N	The MCMC samples.
M	The burn-in samples.
$\text{Var}(.)$	Variance.

Abbreviations

A-D	Anderson-Darling test.
C.I	Confidence interval.
C-V	Cramér-Von test.
CDF	Cumulative hazard function.
EG	Exponentiated generalized.
EGIW	Exponentiated Generalized Inverse Wiebull distribution.
Ge	General entropy loss function.
Is	Importance sampling technique.
iid	independent and identically distributed.
IW	Inverse Weibull distribution.
K-S	Kolmogorov-Smirnov test.
Linex	Linear exponential.
Lx	Linear- exponential loss function.
C.I.L	Lower confidence interval.
MCMC	Markov Chain Monte Carlo .
MLE	Maximum likelihood estimate.
MSE	Mean squared error.
MH	Metropolis-Hastings technique.
Pdf	Probability density function.
QLD	Quasi Lindley Distribution.
Se	Squared error loss function.
C.I.U	Upper confidence interval.

CHAPTER 1

INTRODUCTION

This chapter represents a brief review about some definitions and concepts of reliability,stress-strength reliability, multi-component stress-strength reliability. Also, we will reviewsome methods of estimation that will be used in our thesis. In addition, we propose some Monte Carlo techniques for developing Monte Carlo approximations which willl be used in Bayesian computations.

1.1Reliability Concepts and Principles

Meeker and Escobar (1998) have defined the reliability as; the probability that a component, part, equipment, or system will satisfactorily perform its intended function under given circumstances, such as (environmental conditions, limitations as to operating time, frequency and thoroughness of maintenance), for a specified period of time.

According to this definition, the basic elements of reliability are probability, adequate performance, duration of adequate performance and operating conditions. The above definition covers all four aspects of product, unlike quality, which speaks only according to specifications. In other words reliability is quality over time, which is under the influence of time and environment. Unlike quality, which is a degree of confirmation alone not considering the time length and environment of operation.

Another important difference between quality and reliability is that one can manufacture reliable systems using less reliable components. This by altering product configuration, whereas it is not possible to manufacture high quality systems with less quality components. Adding one or more similar components in parallel can increase the reliability of the system.

Any system will be absolutely reliable of some undesirable events, called failures, do not occur in the system's operation. A failure is the partial or total loss or change in the properties of a device in such a way that its functioning is seriously affected or totally stopped. Every system has its own set of such undesirable events. For example, a failure of a watch may be defined as a delay exceeding 5 sec over a 24-h period.

For a mechanical system, a failure is a breakdown (a crack) of some of its parts or an increase in vibration above the permitted level, etc. One of the most dangerous failures of a nuclear reactor is a leak of a radioactive material. For a missile, the failure could mean missing the target or exploding before hitting it.

1.1.1Reliability function(Survival function)

We first examine reliability as a function of time, and this leads to the definition of hazard rate, which is a very important concept in reliability work. Examining the time dependence of hazard rates allows us to gain insight into the study of failures. This characteristic is very useful in the nature of reliability. Similarly, the time dependence of failures can be viewed in terms of failure modes to differentiate between failures caused by different mechanisms.

Reliability can be expressed in terms of the time to failure T, as following:

$$R(t) = P(T > t) = 1 - F(t). \tag{1.1}$$

Thus, reliability is the probability of no failures (survive) in the interval $[0,t]$ or, in other words, the probability of failure after time t. However, most of the time T will be a continuous random variable and its distribution $F(t)$ will be a continuous distribution having a density function $f(t)$, so the reliability can be written as:

$$R(t) = l - F(t)$$
$$= 1 - \int_0^t f(t)dt$$
$$= \int_t^\infty f(t)dt. \quad (1.2)$$

Figure 1.1 presents the relationship between $f(t)$, $F(t)$, and $R(t)$ graphically.

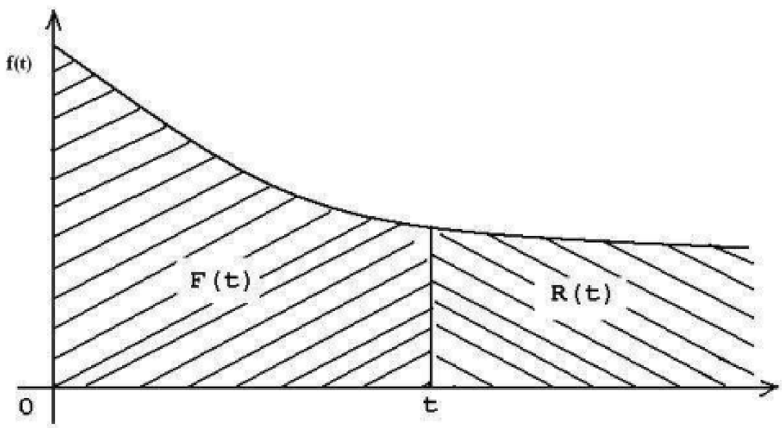

Figure 1.1: Relationship between $f(t)$, $F(t)$, and $R(t)$

1.1.2 Failure rate function (Hazard function)

Sometimes it is difficult to assign the distribution function of T directly from the physical information that is available. A useful function in clarifying the relationship between physical modes of failure and the probability distribution of T is the conditional density function $h(t)$, which called the hazard function or failure rate function.

The hazard function is defined as the instantaneous conditional probability of failure in a small interval of time $(t + \Delta t)$ divided by the width of the interval.

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t < T < t + \Delta t / T > t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{P(t < T < t + \Delta t)}{P(T > t)\, \Delta t}. \qquad (1.3)$$

Since,

$$R(t) = P(T > t),$$

and

$$P(t < T < t + \Delta t) = P(T < t + \Delta t) - P(T < t)$$
$$= R(t) - R(t + \Delta t).$$

Then the hazard function $h(t)$ becomes,

$$h(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t + \Delta t)}{\Delta t} \frac{1}{R(t)}$$
$$= \frac{1}{R(t)} [\frac{-d}{dt} R(t)]$$
$$= \frac{f(t)}{R(t)}. \qquad (1.4)$$

In some situations there is interest in a function called cumulative hazard function.

$$H(t) = \int_0^t h(t) dt$$
$$= \int_0^t \frac{f(t)}{1 - F(t)} dt. \qquad (1.5)$$

It is seen that
$$h(t) = \frac{-R'(t)}{R(t)}$$
$$= \frac{-d(\ln R(t))}{dt}$$

then, $R(t) = e^{-H(t)}$ and $H(t) = -\ln R(t)$ (1.6)

Thus the condition that $R(t) \leq 1$ indicates that $H(t) \geq 0$. The cumulative hazard has been proposed as an effective characteristic to use as a basis for the determination of the failure distribution through the use of plotting techniques (see, for example, Nelson (1972) or Nelson (1982)).

1.2 Stress-Strength Reliability Model

Stress-strength reliability model is one of the most important models that measure the reliability of the product. The term "stress" mean any applied load or load-induced response quantity that has the potential to cause failure. The stresses that cause the failure mechanism can be mechanical (as; deformation, fracture, rupture), electrical(as; electrostatic discharge, dielectric breakdown, junction breakdown in semiconductor devices, hot electron injection, surface and bulk trapping, surface breakdown), thermal(as; heating temperature, thermal expansions and contractions), radiation(as; radioactive containment, secondary rays), and/or chemical(as; corrosion, oxidation).

Often an item failure can be the result of interactions among these various types of stresses. Temperature has a strong effect on the failure of electronic components. Lall (1996) discussed the effect of temperature on the reliability of microelectronics. The term "strength" mean the ability of the component or system to withstand the applied load ("stress").

It is a well accepted fact that the strength of a manufactured unit is a variable quantity that should be modeled as a random variable. This fact forms the basis for all reliability modeling. A second source of variability may also have to be taken into account, when checking the reliability of equipment on the viability of a material, it is also necessary to take into account the stress conditions of the operating environment.

That is, uncertainty about the actual environmental stress to be encountered should be modeled as random. The expression stress-strength model makes explicit that both stress and strength are treated as random variables.

If X is the strength of a system (or component) which is working under a stress Y, both X and Y are generally assumed to be random variables, then stress-strength reliability R of the system is defined as:

$$R = P(Y < X). \tag{1.7}$$

Assume X, and Y are statistically independent random variables with pdf $f(x)$ and $g(y)$, respectively, then the stress-strength reliability can be obtained as:

$$R = P(Y < X).$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x,y) \; dydx.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x)g(y) \; dydx.$$

$$= \int_{-\infty}^{\infty} f(x)G_y(x) \; dx. \tag{1.8}$$

1.3 Reliability for Multi-Component Stress-Strength Model

Several methods exist to improve the system reliability like using large safety factors, reducing the complexity of the system, increasing the reliability of the components, etc. Reliability of a system can be improved by adding one or more similar components in a certain configuration. There are several types of configurations available, such as, series configuration, parallel configuration, mixed configuration, series-parallel configuration, parallel-series configuration, $s-\text{out}-\text{of}-k$ configuration, and others.

Bhattacharya and Johnson (1974) have suggested a system consisting of k identical components and defined a multi-component $s-\text{out}-\text{of}-k$ stress-strength model. Bhattacharya and Johnson (1975) study the condition where a system, consist of k components, functions when at least $s-\text{out}-\text{of}-k$ components survive a common chock of a random magnitude.

Let the random samples $Y, X_1, X_2, \cdots, X_k$ be independent, $G(y)$ be the cumulative distribution function of stress Y and $F(x)$ be the common cumulative distribution function of strengths $X_1, X_2, \cdots, X_k$. The reliability for a multi-component stress-strength model is given by:

$$R_{s,k} = P[\text{at least } s \text{ of the } (X_1, X_2, \cdots, X_k) \text{ exceed } Y]$$

$$= \sum_{i=s}^{k} \binom{k}{i} \int_0^{\infty} [1-F(y)]^i [F(y)]^{k-i} dG(y). \qquad (1.9)$$

Stress-strength reliability is estimated to obtain the reliability function of the parameters for each used distribution. Two methods of estimation are used maximum likelihood and Bayesian estimations.

1.4 Some Applications of Stress-Strength Models

In their landmark book on stress-strength models, Kotz et.al (2003) detail many examples of stress-strength models in a survey of scientific literature. These include such applications as:

Reliability of rocket engines:

When X is the strength of a rocket chamber and Y stands for the maximal chamber pressure which is generated when a solid propellent is ignited, $P(Y < X)$ is the probability that the engine will be fired successfully.

Earthquake Resistance:

The strength stress model was used to study the risk of an earthquake posed to a particular nuclear generator. With no concrete numbers to define the strength, the researcher took strength estimates from five experts and used the log-normal distribution as a model and a weighted least squares procedure to estimate the strength. A similar procedure was used for the stressor, and the conclusion $P(lnY < lnX) = 0.99978$ was reached a very reassuring number, if accurate.

In a medical study:

The reaction of leprosy patients to a medicine was modeled on a $P(Y < X)$ stress-strength model. Initial condition (infiltration status) was taken as X, and Y the change in health after 48 weeks of treatment. The null hypothesis, that initial infiltration values did not affect outcomes, was strongly rejected after an analysis of the data.

1.5 Methods of Estimation

Estimation is one of the important problems in statistical inference, that using a sample of data to guess or estimate the characteristics(parameters) for a population model from which the data are assumed to arise. When you want to determine information about a particular population characteristic (for example, the mean), you usually take a random sample from that population because it is difficult to measure the entire population. Using that sample, you calculate the corresponding sample characteristic, which is used to summarize information about the unknown population characteristic.

The population characteristic of interest is called a parameter and the corresponding sample characteristic is the sample statistic or parameter estimate. Because the statistic is a summary of information about a parameter obtained from the sample, the value of a statistic depends on the particular sample that was drawn from the population. Its values change randomly from one random sample to the next one, therefore a statistic is a random quantity(variable).

The objective of statistical estimation is to assign numerical values to the parameter based on the sample data. There are two main methods of estimation point estimation and interval (or confidence interval) estimation.

1.5.1 Point estimation

Point estimation is to estimate one value for the unknown parameter from the desired distribution to choose an estimator. A number of properties that evaluate the performance of the procedure in the context of the assumed distribution function are considered. We look at a few of these in the next subsection.

Properties of best estimator

Now, we define some basic properties that must be satisfied for the point estimator to be good.

1- **Unbiased**.

A point estimator is unbiased for a parameter if the mean(expectation) of the estimator's sampling distribution equals the value of the parameter; i.e. $E(\hat{\theta}) = \theta$, otherwise, the estimator is biased.

2- **Minimum MSE**.

The mean square error(MSE) of $\hat{\theta}$(the estimator of θ) is the expected value of $(\hat{\theta} - \theta)^2$.

$$MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2 \tag{1.10}$$

MSE is a measure to the goodness of a point estimator, it is always non-negative, and values closer to zero are better.

3- **Consistency**.

An estimator $\hat{\theta}$ of θ, is said to be consistent if for any $\varepsilon > 0$ and all possible values of θ, $P(|\hat{\theta}-\theta|>\varepsilon)\to 0$ as $n\to\infty$.Then for a consistent estimator the probability that the estimate will deviate from the true value by any amount, no matter how small, approaches zero as the sample size becomes increasingly large; i.e., for large samples the estimate will be very close to the true parameter value with high probability.

4-**Sufficiency**.

An estimator $\hat{\theta}$ ،of θ is sufficient if the conditional distribution of the random sample $X_1, X_2, \cdots, X_n$ given θ does not depend on θ. This implies that the estimator contains all information in the sample about the parameter. It is sufficient to know its value; given that, no additional information about θ is contained in the data.

5- **Efficiency**.

The efficiency of an estimator is measured in terms of its variability. The rationale is that use of an inefficient estimator requires more data to do as well and hence it costs less to use an efficient estimator. Efficiency of an estimator may be assessed relative to another estimator or estimators relative efficiency or relative to an absolute standard.

Let $\hat{\theta}_1$ and $\hat{\theta}_2$ are two unbiased estimators of θ, then $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$ if $Var(\hat{\theta}_1) < Var(\hat{\theta}_2)$.

6-**Minimum Variance Unbiased Estimator (MVUE)**.

An unbiased estimator $\hat{\theta}$ of a parameter θ, is said to be minimum variance unbiased if$Var(\hat{\theta}) \le Var(\theta^*)$ for any other unbiased estimator θ^* and for all possible values of θ.Rao-Blackwell theorem is used to find the uniformly minimum variance unbiased estimator(UMVUE).

Theorem 1.1 (Rao-Blackwell Theorem)

Let θ^* an estimator of θ with $E(\hat{\theta}^2)$ for all $\theta < \infty$. Suppose that T is a sufficient statistic for θ, and let $\hat{\theta} = E(\hat{\theta}/T)$ is a function of the sufficient statistic for θ, then for all θ:

$$E(\hat{\theta}-\theta)^2 \le E(\theta^* - \theta)^2.$$

We now have a quantitative rationale for basing estimators on sufficient statistics: if an estimator is not a function of a sufficient statistic, then there is another estimator which is a function of the sufficient statistic and which is at least as good, in the sense of mean squared error of estimation.

1.5.2 Maximum likelihood estimation

Given $X_1, X_2, \cdots, X_n$ an iid sample with probability density function $f(x_i;\Theta)$, where $\underline{\Theta}$ is a $(k \times 1)$ vector of parameters that characterize $f(x_i;\underline{\Theta})$. The joint density of the sample is, by independence, equal to the product of the marginal densities:

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &= f(x_1;\underline{\Theta}) f(x_2;\underline{\Theta}) \cdots f(x_n;\underline{\Theta}) \\ &= \prod_{i=1}^{n} f(x_i;\underline{\Theta}). \end{aligned} \tag{1.11}$$

The likelihood function is defined as the joint density treated as a functions of the parameters $\underline{\Theta}$.

$$\begin{aligned} L(\underline{\Theta}/x_1, x_2, \cdots, x_n) &= f(x_1, x_2, \cdots, x_n) \\ &= \prod_{i=1}^{n} f(x_i;\underline{\Theta}). \end{aligned} \tag{1.12}$$

The maximum likelihood estimator (MLE) denoted by $\underline{\Theta}_{mle}$, are the values of $\underline{\Theta}$ that maximizes the likelihood function $L(\underline{\Theta}/x_i)$.

$$\underline{\Theta}_{mle} = arg \;\; max_{\underline{\Theta}} \; L(\underline{\Theta}/x_i), \tag{1.13}$$

where

$\underline{\Theta} = \{\theta_1, \theta_2, \cdots, \theta_k\}$.

The maximum likelihood estimator(MLE) of the parameters $\underline{\Theta}$ is obtained by differentiation of the likelihood function $L(\underline{\Theta}/x_i)$ with respect to these parameters and equal to zero. In practice, when finding the maximum likelihood estimator, it is often easier to find the value of the parameter that maximize the natural logarithm of the likelihood function $\ell(\underline{\Theta}/x_i)$ rather than the value of the parameter that maximize the likelihood function itself.

Because the natural logarithm is an increasing function, the solution will be the same. If $\ell(\underline{\Theta}/x_i)$ is differentiable with respectto the parameters, we can find the Ml estimator $\hat{\underline{\Theta}}$ of $\underline{\Theta}$ as a solution of the system of equations;

$$\frac{\partial log L(\underline{\Theta}/x_i)}{\partial \underline{\Theta}} = \frac{\partial \ell(\underline{\Theta}/x_i)}{\partial \underline{\Theta}} = 0. \tag{1.14}$$

The obtained solutions are necessary critical points (maximum, minimum, or saddle point) of the log-likelihood function, To actually prove that the solution is a maximum, we need to show in scalar case $\frac{d^2\ell(\theta)}{d\theta^2} < 0$ for one parameter, or if θ is a vector that the Hessian matrix $H(\theta)$ defined by $H(\theta) = \left(\frac{\partial^2 \ell(\theta)}{\partial\theta_i \partial\theta_j}\right)_{k\times k}$, $i, j = 1, 2, ..., k$ is negative definite.

The equation $\frac{d\ell(\theta)}{d\theta} = 0$ for scalar θ or the system of equations $\frac{\partial}{\partial\theta_j}\ell(\theta) = 0$, for vector θ has a unique root $\hat{\theta} \in (-\infty, \infty)$, if and only if,

$J(-\infty) > 0$ and $J(\infty) < 0$, where $J(\theta) = \frac{\partial}{\partial\theta}\ell(\theta)$.

If $\ell(\theta)$ has multiple local maxima, we pick the solution that is the highest of all the maximizers. For further information see Ghitany et al. (2014) and El-Din et al. (2017).

The method of maximum likelihood is the most popular technique for deriving estimators, it find an estimation of the unknown parameter that give the max probability of the observed data. In our research, we not only estimate the parameters which is the characteristics of an distribution, but also estimate a function of these parameter.

Stress-strength reliability estimation is our problem, as shown in Eq.(1.8) that R is a function of the prameters of the probability density functions $f(x)$ and $g(y)$, so the maximum likelihood function will be equal to the product of the marginal densities.

Suppose that $X_1, X_2, \cdots, X_n$ are random samples from distribution , and $Y_1, Y_2, \cdots, Y_m$ are random samples from another distribution, and X, Y are independent and identically (iid) random samples, then likelihood function can be written as:

$$L(\underline{\Theta}/x, y) = \prod_{i=1}^{n} f(x) \prod_{j=1}^{m} g(y), \tag{1.15}$$

where $\underline{\Theta}$ is the estimated parameters which arises in $f(x)$ and $g(y)$.

Definition: Invariance property of maximum likelihood estimators

One of the attractive features of the method of maximum likelihood is its invariance to one-to-one transformations of the parameters of the log-likelihood. That is, if $\hat{\theta}$ is the MLE of θ and $g = h(\theta)$ is a one-to-one function of θ then $\hat{g} = h(\hat{\theta})$ is the MLE for g.Other estimates do not possess such an invariance property, like Bayes estimates.

Asymptotic Properties of Maximum Likelihood Estimators

Much of the interest of maximum likelihood estimators is based on their properties for large sample sizes. We summarizes some the important properties below;

1.**Consistency**

The estimate $\hat{\theta}$ is called consistent if:

$$\hat{\theta} \to \theta_0 \text{ in probability as } n \to \infty,$$

where θ_0 is the true unknown parameter of the distribution of the sample. In words, as the number of observations increase, the distribution of the maximum likelihood estimator becomes more and more concentrated about the true state of nature.

2.**Asymptotic normality**

Using the Central limit theorem; we say that $\hat{\theta}$ is asymptotically normal if:

$$(\sqrt{n}(\hat{\theta} - \theta) \overset{d}{\to} N(0, I^{-1}(\Theta)). \text{ as } n \to \infty,$$

where $\overset{d}{\to}$ means converge in distribution, and $I^{-1}(\underline{\Theta})$ is the inverse of the Fisher information matrix $I(\underline{\Theta})$.

The Fisher information Matrix

First we define the Hessian matrix, which is a $k \times k$ symmetric matrix whose element is given by of second derivatives of the log-likelihood function $l(\underline{\Theta}/X)$.

$$H(\underline{\Theta}/X)=\begin{pmatrix} (\frac{\partial^2 l}{\partial^2\theta_1}) & (\frac{\partial^2 l}{\partial\theta_1\partial\theta_2}) & \cdots & (\frac{\partial^2 l}{\partial\theta_1\partial\theta_k}) \\ (\frac{\partial^2 l}{\partial\theta_2\partial\theta_1}) & (\frac{\partial^2 l}{\partial^2\theta_2}) & \cdots & (\frac{\partial^2 l}{\partial\theta_2\partial\theta_k}) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ E(\frac{\partial^2 l}{\partial\theta_k\partial\theta_1}) & (\frac{\partial^2 l}{\partial\theta_k\partial\theta_2}) & \cdots & (\frac{\partial^2 l}{\partial^2\theta_k}) \end{pmatrix}. \tag{1.16}$$

The Fisher information matrix is defined as minus the expectation of the Hessian matrix:

$$I(\underline{\Theta}/X)=-E(H(\underline{\Theta}/X)). \tag{1.17}$$

This definition corresponds to the expected Fisher information matrix. If no expectation is taken we obtain a data-dependent quantity that is called the observed Fisher information. Fisher information matrix essentially describes the amount of information data provide about the unknown parameters. It used in finding the variance of an estimator, as well as in the asymptotic behavior of maximum likelihood estimates.

1.5.3Interval Estimation

In point estimation of a parameter or other population characteristic, we use a single number to estimate a parameter or a set of k numbers to estimate a k-dimensional parameters. For MLEs, we also gave MSE of the estimators, which is a measure of uncertainty in the estimate. A confidence interval (or confidence interval estimator) takes this uncertainty into account by providing an estimate in the form an interval of numbers along with a measure of the "confidence" one has that the interval will, in fact, contain the true value of the parameter or characteristic being estimated. For k parameters, a separate confidence interval may be calculated for each, or a k dimensional confidence region may be defined.

In the case of a single parameter, say θ, a confidence interval based on a sample of size n, $X_1, X_2, \cdots, X_n$ is defined as an interval defined by two limits, the lower limit $L_1(X_1, X_2, \cdots, X_n)$ and the upper limit $L_2(X_1, X_2, \cdots, X_n)$, having the property that

$$L_1(X_1, X_2, \cdots, X_n) \leq \theta \leq L_2(X_1, X_2, \cdots, X_n) = \gamma, \qquad (1.18)$$

where γ (the confidence coefficient) is a constant with $0 < \gamma < 1$.

Then the confidence coefficients is the probability that the interval estimate will contain the parameter. Confidence is usually expressed in percent; e.g., if $\gamma = 0.95$, the result is a "95%" confidence interval" for θ. It is desirable in practice that the width of a confidence interval be small, i.e., that the result be precise in the sense that we can have high confidence that the true value of the parameter lies within a relatively narrow interval (or small region, in the multi-parameter case).

In general, the width of the interval depends on the data and on the desired confidence. The width of the confidence interval has the following properties:

- Decreases as n increases.
- Increases as the confidence coefficient γ increases.
- Decreases as the variability in the data decreases.

Thus in theory, the width of the confidence interval can be controlled, but in practice this is not always easy. In particular, it usually means incurring the expense of obtaining large samples.

There are a lot of methods for interval estimation such as; Asymptotic confidence interval using the Fisher information matrix, HPD interval(highest posterior density), and bootstrap confidence interval, see Efron's (2003).

1.5.4Bayesian Estimation

Bayesian inference grows out of the simple formula known as Bayes rule. Assume we have two random variables A and B. A principle rule of probability theory known as the chain rule allows us to specify the joint probability of A and B taking on particular values a and b which gives us:

Joint probability $=$ Conditional Probability $\times$ Marginal Probability.

Thus we have:

$$P(a,b) = P(a \mid b)P(b) \tag{1.19}$$

There is nothing special about our choice to marginalize B rather than A, and thus equally we have:

$$P(a,b) = P(b \mid a)P(a) \tag{1.20}$$

When combining the two equations (1.19),(1.20) we get:

$$P(a \mid b)P(b) = P(b \mid a)P(a) \tag{1.21}$$

rearranged as:

$$P(a \mid b) = \frac{P(b \mid a)P(a)}{P(b)}. \tag{1.22}$$

and can be equally written in a marginalized form as:

$$P(a \mid b) = \frac{P(b \mid a)P(a)}{\int P(b \mid a')P(a')da'}. \tag{1.23}$$

This expression is Bayes Rule, which indicates that we can compute the conditional probability of a variable A given the variable B from the conditional probability of B given A. This introduces the notion of prior and posterior knowledge.

A prior probability P(a) is the probability available to us beforehand, and before making any additional observations. A posterior probability $P(a \mid b)$ is the probability obtained from the prior probability after making additional observation to the prior knowledge available. The additional observation was observing that B takes on value b. When dealing with parameter estimation, θ could be a parameter needed to be estimated from some given evidence or data. The probability of data given the parameter is commonly referred to as the likelihood $L(data \mid \theta)$. And so, we can compute the probability of a parameter given the likelihood of some data, which called the posterior function.

$$\pi(\theta \mid data) = \frac{L(data \mid \theta)\pi(\theta)}{\int L(data \mid \theta')\pi(\theta')d\theta'}. \quad (1.24)$$

Thus the inference concerning θ is then based on its posterior function $\pi(\theta \mid data)$.

1.5.4.1 Prior distributions

An important problem in Bayesian analysis is how to define the prior distribution. For prior distributions in Bayesian inference, the most used priors are conjugate and non informative priors, described as following:

a- Conjugate priors

A prior is said to be a conjugate prior when the prior and the posterior belong to the same distribution family. For example in the case of a binomial likelihood we have just seen that any beta prior we use will result in a posterior that is also a beta distribution. In this case the beta distribution is a conjugate prior for the Binomial likelihood. Conjugate priors are very useful as they provide simple analytic solution to this type of inference problem, but they are also somewhat limiting since our prior belief may not be representable using the conjugate family's parameterization.

b- **Non-informative priors**

Non-informative priors are used when relatively little information is available about prior sampling of the parameter θ, thus, a prior is non-informative if it has minimal impact on the posterior distribution of θ. The uniform distribution is frequently used as a non-informative prior. In some cases, non-informative priors can lead to improper posteriors (non integrable posterior density). You cannot make inferences with improper posterior distributions .

1.5.4.2 Loss function

Bayesian estimation is a special case of decision rule that minimizes the expected loss value, to achieve a minimum probability of error. Consider $\hat{\theta}$ is an estimator of θ, loss function $Ls(\hat{\theta},\theta)$ is used as a measure of error, it is defined as a real-valued function that satisfying:

- $Ls(\hat{\theta},\theta) \geq 0$ for all possible estimators $\hat{\theta}$ and all $\theta \in \underline{\Theta}$.
- $Ls(\hat{\theta},\theta) = 0$ for $\hat{\theta} = \theta$.

We obtain a Bayes estimate, $\hat{\theta}$ of the parameter θ by choosing a particular form of loss function, $Ls(\hat{\theta},\theta)$. To obtain the Bayes estimate first we need to find the posterior expected loss $E(Ls(\hat{\theta},\theta))$ by $\int_{\theta} Ls(\hat{\theta},\theta)\,\pi(\theta \mid data)\,d\theta$ which is also known as the posterior risk for θ.

Then we minimize it with respect to $\hat{\theta}$. It is to be noted that different Bayes estimates of θ will be obtained depending on the different loss functions.

Depending on the complexity of the loss function L and the posterior distribution $\pi(\theta \mid data)$, the value of $\hat{\theta}$ may be determined analytically or numerically.

In general, it is difficult to determine the value of $\hat{\theta}$ analytically because of either the complicated posterior distribution or the complex loss functions. Nevertheless, there are some loss functions for which the analytical Bayes estimates are feasible.

Three different types of loss functions are used in the next chapters, described as following:

1-**Squared error loss function (Se)**:

It is a symmetric function given by

$$Ls_{Se}(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2 \qquad (1.25)$$

Under the squared error loss function (Se), the Bayes estimate for θ is the posterior mean which given by:

$$\hat{\theta}_{Se} = E(\theta).$$

$$= \int_{\theta} \theta \; \pi(\theta \mid data) d\theta. (1.26)$$

2-**Linex loss function (Lx)**:

A very useful asymmetric loss function, introduced by Varian (1975), which mean linear-exponential loss function. It become approximately linear to one side of the origin, and approximately exponential to the other side.

$$Ls_{Lx}(\hat{\theta},\theta) = exp[c(\hat{\theta}-\theta)] - c(\hat{\theta}-\theta) - 1. \qquad (1.27)$$

Where c is constant, $c \neq 0$. The sign and the magnitude of c represent the direction and the degree of asymmetry, respectively.

The Bayesian estimate under the linex loss function (Lx) is given by:

$$\hat{\theta}_{Lx} = \frac{-1}{c} ln\{E(exp(-c\,\theta))\}, \tag{1.28}$$

provided that the expectation $E(exp(-c\,\theta))$ is finite. Several papers have applied this loss function with different value for the constant c as; Zellner (1986), Basu and Ebrahimi (1991), Soliman (2000) and Parsian and Kirmani(2002).

3-**General entropy loss function (Ge)**:

This loss function is asymmetric loss function,given by :

$$Ls_{Ge}(\hat{\theta},\theta) \propto (\frac{\hat{\theta}}{\theta})^{q} - q\, ln(\frac{\hat{\theta}}{\theta}) - 1. \tag{1.29}$$

The Bayes estimate relative to the general entropy loss function (Ge) is given by:

$$\hat{\theta}_{Ge} = \left\{E(\theta)^{-q}\right\}^{-1/q}. \tag{1.30}$$

where q is constant.

For $q=1$ the Bayesian estimate with (Ge) become the Bayesian estimate under the squared error loss function. Calabria and Pulcini (1994) used this function for a different values of q. Pandey and Rao (2009), and SankuDey (2010) have used this loss for Bayesian estimation.

1.5.5 Monte Carlo methods for Bayesian computations

Monte Carlo methods is a class of computational algorithms that depend on repeated random sampling to obtain numerical approximations, often they used for evaluating complex integrals; see Smith(1991). Monte Carlo methods are based on random samples generated from a density related to a parameter of interest, which denoted by the posterior function in Bayesian estimate. The most popular method to do this today is the Markov Chain Monte Carlo (MCMC) method.

MCMC is a class of methods for sampling a pdf using a Markov chain whose equilibrium distribution is the desired distribution. Once we have a sample distributed according to some desired distribution, we can compute expectation values and integrals of various quantities in a process analogous to Monte Carlo integration.

For example; Combining Eq. (1.24), and Eq. (1.26) to get Bayesian estimate of θ under squared error loss function, then we have the following equation:

$$\hat{\theta}_{Se} = \int_{\theta} \theta\, \pi(\theta \mid data) d\theta.$$

$$= \frac{\int_{\theta} \theta\, L(data \mid \theta)\pi(\theta) d\theta}{\int_{\theta} L(data \mid \theta)\pi(\theta) d\theta}, \qquad (1.31)$$

where $L(data \mid \theta)$ is the likelihood function, and $\pi(\theta)$ is the prior function.

The explicit evaluation for Eq. (1.31) is not possible, and become more difficult for the high dimensional parameters. Monte Carlo method provides a technique where we can sample from the posterior directly, then obtain sample estimate of this integral, thus we can perform the integration in implicit form. Some methods of Monte Carlo technique have been introduced as the following:

1.5.5.1 Gibbs sampling

The main basis MCMC method is Gibbs sampling, which is a special case of the Metropolis-Hastings (MH) algorithm that is very simple to use in practice. As discussed in Besag and Green (1993), the Gibbs sampler is founded on the ideas of Grenander (1983), while the formal term is introduced by Geman and Geman (1984). The primary bibliographical landmark for Gibbs sampling in problems of Bayesian inference is Gelfand and Smith (1990). Suppose we have a set of k parameter vectors; $\underline{\Theta} = \theta_1, \theta_2, \cdots, \theta_k$, where each θ_k could be a scalar or a vector of parameter, and let $\pi(\underline{\Theta}/data)$ be its posterior distribution given the data. Then, the basic algorithm of the Gibbs sampler is given as follows:

- Step1: Choose an arbitrary starting value of $\underline{\Theta}^i$, and set $i = 0$.
- Step2: Generate $\theta_1^{(i+1)}$ from $\pi(\theta_1/\theta_2^{\,i}, \cdots, \theta_k^{\,i}, data)$.
- Step3: Generate $\theta_2^{(i+1)}$ from $\pi(\theta_2/\theta_1^{\,i+1}, \theta_3^{\,i}, \cdots, \theta_k^{\,i}, data)$.
- ....
- Step4: Generate $\theta_k^{(i+1)}$ from $\pi(\theta_k/\theta_1^{\,i+1}, \theta_2^{\,i+1}, \cdots, \theta_{k-1}^{\,i+1}, data)$.
- Step5: Set $i = i + 1$, and go to Step 1.

The above conditional distributions are the transition distributions of a Markov chain that converges (under very general conditions) to a unique stationary target distribution that is the posterior distribution $\pi(\underline{\Theta}/data)$. The generic Gibbs sampler algorithm is to draw one value for each $\underline{\Theta}^i$ from its conditional distribution and cycle through these conditionals repeatedly. This approximation can be made arbitrarily accurate by increasing the sample size, k. Given that it is now computationally inexpensive to obtain tens of thousands of draws on any standard computer for all but the most complex and highly dimensional models, Gibbs sampling is an easy way to draw posterior inferences concerning any unknown quantities in a model.

1.5.5.2 Metropolis Hasting algorithm (MH)

The most general MCMC algorithm is the Metropolis-Hastings(MH) algorithm, which was originally introduced by Metropolis et al.(1953), and subsequently generalized by Hastings (1970). Tierney(1994) gives a comprehensive theoretical presentation of this algorithm, and Chib and Greenberg(1995)introduced an excellent tutorial on this topic. The Metropolis-Hastings (MH) algorithm simulates samples from a probability distribution by making use of the full joint density function and (independent) proposal distributions for each of the variables of interest.

Suppose we are interested in sampling from the posterior distribution $\pi(\theta/\text{data})$, MH algorithm uses a two step process:

- Specify a proposal distribution $q(\theta,\vartheta)$.
- Accept draws from $q(\theta,\vartheta)$ with acceptance ratio

$$\alpha(\theta_i,\theta^{å}) = min\left[\frac{\pi(\vartheta/D)\, q(\vartheta,\theta)}{\pi(\theta/D)\, q(\theta,\vartheta)},1\right].$$

Also let $U(0,1)$ denote the uniform distribution over $(0,1)$, The Metropolis-Hastings algorithm for sampling from the posterior distribution $\pi(\theta/D)$ can be described as follows:

- Step1: Choose an arbitrary starting value of θ_0, and set $i = 0$.
- Step2: Generate a candidate point $\theta^{å}$ from $q(\theta_i,...)$ and u from U(0,1).
- Step3: Set $\theta_{i+1} = \theta^{å}$ if $u \le \alpha(\theta_i,\theta^{å})$, otherwise $\theta_{i+1} = \theta_i$.
- Step5: Set $i = i+1$, and go to Step 1.

The above algorithm is very general. When $q(\vartheta,\theta) = q(\vartheta)$, the Metropolis-Hastings algorithm reduces to the independence chain Metropolis algorithm(see Tierney 1994). The Gibbs sampler can also be shown to be a special case of the MH algorithm that uses conditional distributions as proposal distributions with acceptance probability always equal to 1 ($\alpha(\theta_i, \theta^{å}) = 1$); see Geyer (2011).

1.5.5.3 Importance Sampling technique (IS)

Importance sampling (IS) refers to a collection of Monte Carlo methods where a mathematical expectation with respect to a target distribution is approximated by a weighted average of random draws from another distribution. Together with Markov Chain Monte Carlo methods, Importance sampling has provided a foundation for simulation-based approaches to numerical integration since its introduction as a variance reduction technique in statistical physics; see; Hammersely and Morton (1954), and Rosenbluth and Rosenbluth (1955). Importance sampling Technique has suggested by Chen and Shao(1999). Nowadays, IS is used in a wide variety of application areas and there have been recent developments involving adaptive versions of the methodology.

The principle idea of the IS estimation can be explained as following; Let $p(x)$ be a probability density for a random variable X and suppose we are interested in computing an expecation μ_f, where:

$$\begin{aligned}\mu_f &= E(f(X)) \\ &= \int f(x)p(x)dx. \qquad (1.32)\end{aligned}$$

Sometimes, it is typically difficult to sample directly from $p(x)$, therefore in practice one usually resorts to drawing from the so called importance density $q(x)$ with the support including the one of the density of interest $p(x)$. It is assumed the sampling from $q(x)$ is relatively easy and inexpensive. This method of simulation based estimation is called importance sampling (IS). Using $q(x)$, the expecation μ_f can be expressed in the following way:

$$\mu_f = \int f(x) \frac{p(x)}{q(x)} q(x) dx.$$

$$= E_q \{ f(X) \frac{p(X)}{q(X)} \}.$$

$$= E_q \{ f(X) w(x) \}. \qquad (1.33)$$

Where E_q stands for expectation with respect to density $q(x)$ and $w(x)$ is known as the importance weight function. Therefore a sample of independent draws $x_1; \cdots ; x_m$ from q(x) can be used to estimate μ_f by

$$\hat{\mu}_f = \frac{1}{m} \sum_{i=1}^{m} f(x_i) w(x_i). (1.34)$$

In many applications the density $p(x)$ is known only up to a normalizing constant. Here one has $w(x) = cw_0(x)$ where $w_0(x)$ can be computed exactly but the multiplicative constant c is unknown. In this case one replaces $\hat{\mu}_f$ with the ratio estimate:

$$\hat{\mu}_f = \frac{\sum_{i=1}^{m} f(x_i) w(x_i)}{\sum_{i=1}^{m} w(x_i)}. (1.35)$$

Importance sampling is widely used in Bayesian computation, see Geweke (1989). This approach provides a focus on an important part of the posterior distribution, which is obtained first, by an appropriate weighting of draws, and second, by generating them from an optimal, tail-focused density. Geweke (1989) also provided guidelines on how to choose a good importance sampling density that has a shape similar to the desired posterior density. It is well-known that using importance sampling, one can easily approximate the posterior expectations.

1.6 Mentioned Statistical Distributions

We present here the two basic distributions that will be used in the subsequent chapters.

1.6.1 Quasi Lindley Listribution (QLD)

The QLD which introduced by Shanker et al. (2013) of which the Lindley distribution is a particular case.

The QLD has a pdf given by:

$$f(x)=\frac{\theta}{\alpha+1}(\alpha+\theta x)\,e^{-\theta x}, \tag{1.36}$$

and CDF:

$$F(x)=1-\left[\frac{(1+\alpha+\theta x)}{\alpha+1}\,e^{-\theta x}\right], \tag{1.37}$$

where $x>0,\theta>0,\alpha>-1.$

1.6.2 Exponentiated Generalized Inverse Weibull Distribution (EGIW)

The EGIW distribution which introduced by Elbatal and Muhammed (2014) as extension of exponentiated generalized family.

The EGIW distribution has a pdf f(x) and CDF F(x):

$$f(x)=\alpha\beta\theta\lambda^{\theta}x^{-\theta-1}e^{-(\frac{\lambda}{x})^{\theta}}\left[1-e^{-(\frac{\lambda}{x})^{\theta}}\right]^{\alpha-1}\left[1-(1-e^{-(\frac{\lambda}{x})^{\theta}})^{\alpha}\right]^{\beta-1}, \tag{1.38}$$

$$F(x)=\left[1-(1-e^{-(\frac{\lambda}{x})^{\theta}})^{\alpha}\right]^{\beta}, \tag{1.39}$$

where $x>0,\lambda,\theta,\alpha,\beta>0.$

1.7 Review of Literature

Several authors have discussed the problem of estimating the stress-strength reliability assuming various lifetime distributions for the stress-strength random variates. The term stress-strength was first introduced by Church and Harris (1970) which introduced the estimation of R when X and Y are normally distributed. Downtown (1973) suggested the minimum variance unbiased estimator of $P(Y < X)$ as alternatives to the asymptotically equivalent estimator used by Church and Harris (1970) to obtain confidence intervals for that probability for the same distribution. Bhattacharya and Johnson (1974) have suggested a system consisting of ‘k’ identical components and introduced a multi-component s out of k stress-strength model.

Tong (1977) discussed the estimation of $P(Y < X)$ for exponential families. Some inference results in $P(Y < X)$ for the bivariate exponential model has been achieved by Awad et al. (1981). Pandey and Borhan (1985) presented the reliability in a multi-component stress-strength system when both stress and strength follow Burr distribution.

Awad and Gharraf (1986) introduced a three estimators for $P(Y < X)$ when Y and X are two independent but not identically distributed Burr random variables. Kakati and Srivastav (1986) studied an accelerated life testing problem for the stress-strength model. The system reliability estimation in multi-component stress-strength systems has been discussed by Pandey and Upadhyay (1986) when stress and strength are Weibull distributions with equal scale parameters.

Gupta and Gupta (1990) considered estimation of $P(X > Y)$ in the multivariate normal case. McCool (1991) examined inference on $P(Y < X)$ in the case of Weibull distribution. Nandi and Aich (1994) have discussed the problem of estimating the reliability $P(Y < X)$ that appears in stress-strength relationship where X follows an exponential distribution while Y has an inverse Gaussian / half normal / half Cauchy distribution.

Pham and Almhana (1995) have presented basic properties of three parameter generalized gamma distribution,also introduced results on the hazard rate and stress-strength model of the generalized gamma distribution.

Inference for $P(Y < X)$ in the Burr type X model has been investigated by Surles and Padgett (1998, 2001). Gupta and Brown (2001) introduced reliability studies of the skew-normal distribution and its application to stress-strength models. A good application on the different stress-strength models can be found in the research by Kotz et al.(2003).

Some of studies on the stress-strength model can be obtained in Kundu and Gupta (2005,2006), Raqab and Kundu (2005), which considered this problem when X and Y are generalized exponential, weibull and Burr type X distributions respectively. The reliability of a stress-strength model with Burr type III distribution has been discussed by Mokhlis (2005).

Kantam et al.(2007) introduced stress-strength reliability model in log-logistic distribution. Krishnamoorthy et al. (2007) introduced an inference on reliability in two-parameter exponential stress-strength model. Raqab et al.(2008) introduced the estimation of $P(Y < X)$ for the three-parameter generalized exponential distribution. Stress-strength reliability for three-parameter Weibull distribution has been discussed by Kundu and Raqab (2009).

Gupta et al.(2010) derived the estimation of reliability from Marshall-Olkin extended lomax distribution. Estimation of stress-strength reliability in multi-component model for log-logistic distribution has been discussed by Srinivasa and Kantam (2010).

Stress-strength reliability for Lindley and weighted Lindley distributions considered by Al-Mutairi et al. (2013), (2015) respectively.

Singh et al.(2014) introduced the estimation of $P(Y < X)$ for generalized Lindley distribution. Khan et al. (2015) studied the estimation of stress-strength reliability model using finite mixture of two parameter lindley distributions.

Recentely; Hanagal and Bhalerao(2016) discussed generalized inverse Weibull software reliability growth model.

Actually, it is impossible to mention here every author who contributed to the development of stress-strength model.

CHAPTER 2

Estimation of Stress-Strength Reliability for the Quasi Lindley Distribution

2.1 Quasi Lindley Distribution (QLD)

As mentioned at the end of chapter 1, Lindley distribution and all distributions that relate to it have been widely used for studies on stress-strength reliability. For example; Al-Mutairi et al.(2013),(2015) presented the stress-strength reliability for Lindley and weighted Lindley distributions respectively. Stress-strength reliability estimation for generalized lindley distribution has been introduced by Singh et al.(2014). Recentely Khan et al. (2015) studied the estimation of stress-strength reliability model using finite mixture of two parameter lindley distributions.

This chapter is focused upon upon studying the problem of the estimation of the stress-strength reliability for the QLD introduced by Shanker et al. (2013) of which the Lindley distribution is a particular case. We will estimate the parameter of the stress-strength reliability R using the maximum likelihood, and Bayesian estimation methods. The asymptotic confidence interval of R will be computed based on the asymptotic distribution of the MLE of R. In Bayesian estimation we will introduce two sampling methods (Importance Sampling and Metrolopis-Hastings).

The QLD has a pdf given by:

$$f(x) = \frac{\theta}{\alpha+1}(\alpha+\theta x)\, e^{-\theta x}, \tag{2.1}$$

and CDF:

$$F(x) = 1 - [\frac{(1+\alpha+\theta x)}{\alpha+1}\, e^{-\theta x}], \tag{2.2}$$

where;

$$x > 0, \theta > 0, \alpha > -1.$$

It can easily be seen that at $\alpha = \theta$, Eq.(2.1) reduces to the pdf of lindley distribution and at $\alpha = 0$, it reduces to the pdf of gamma distribution with parameters $(2, \theta)$.

The graphs of density and distribution functions of QLD for different values of its parameters α and θ are shown in Figure(2.1), (2.2).

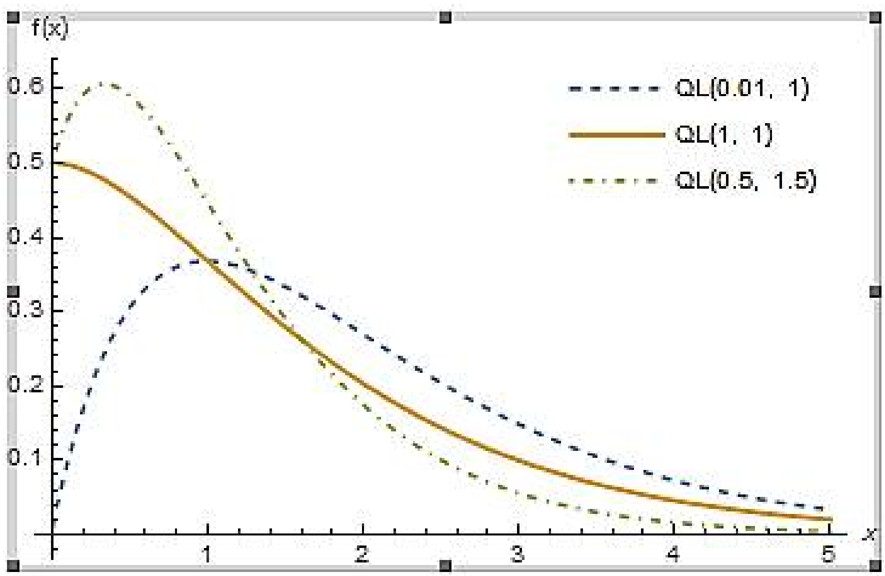

Figure 2.1: Pdf of the QLD for some parameter values α, θ

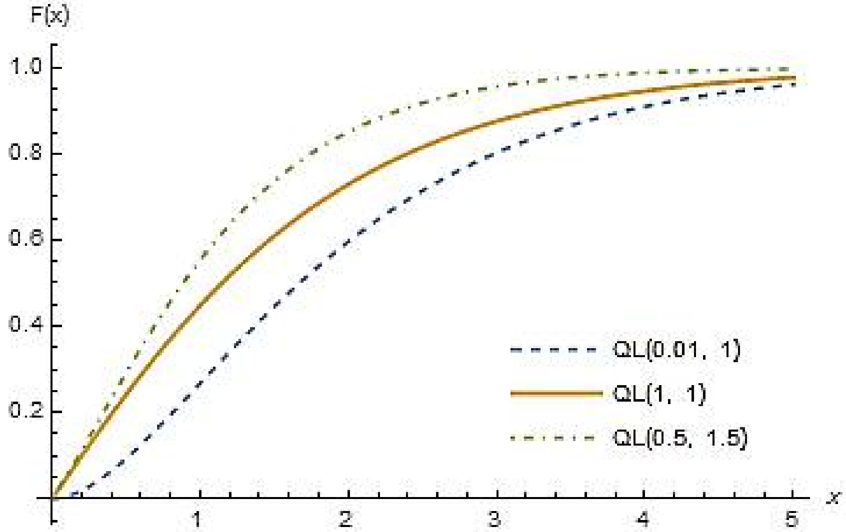

Figure 2.2: CDF of the QLD for some parameter values α, θ

The quasi Lindley distribution has a survival and hazard rate function respectively given by:

$$\begin{aligned} S(x) &= 1 - F(x) \\ &= \frac{(1+\alpha+\theta x)}{(\alpha+1)} e^{-\theta x}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} h(x) &= \frac{f(x)}{1-F(x)} \\ &= \frac{\theta((\alpha+\theta x))}{(1+\alpha+\theta x)}. \end{aligned} \tag{2.4}$$

Figures 2.3 and 2.4 illustrate survival and hazard (failure) rate functions of QLD for selected values of the parameters.

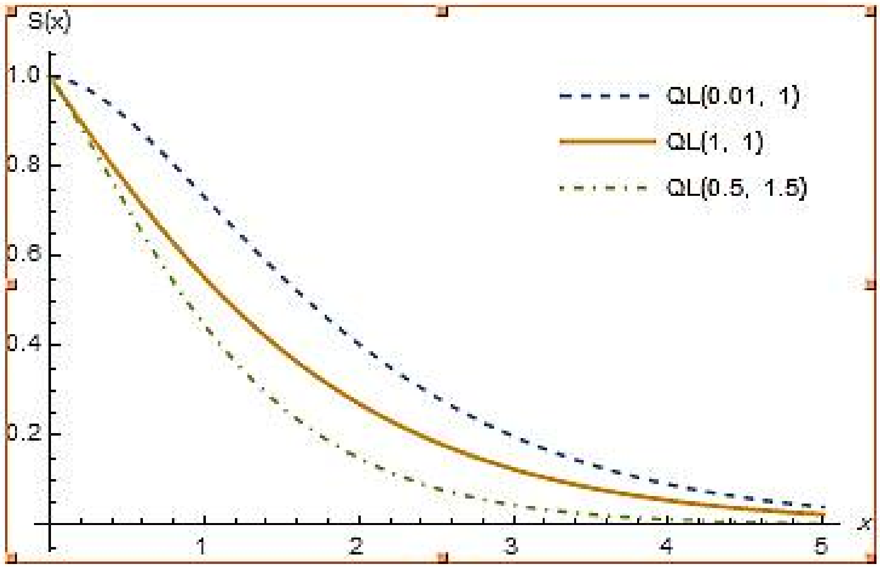

Figure 2.3: The survival function of the QLD for some parameter values α, θ

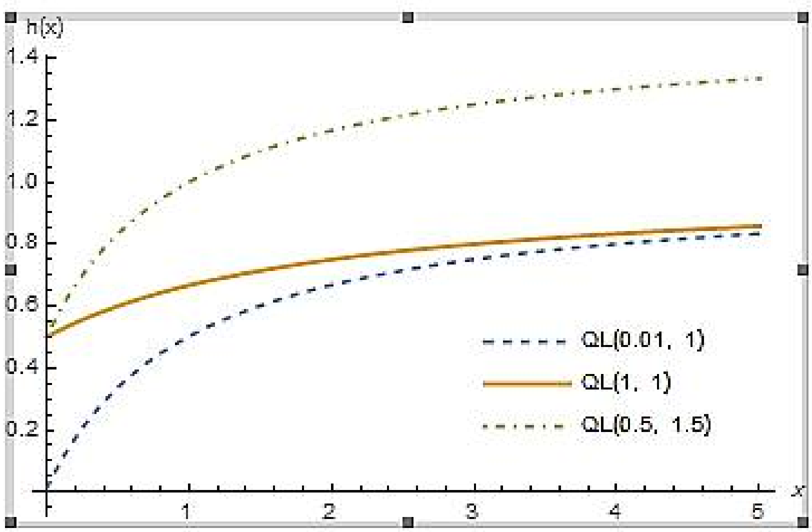

Figure 2.4: The hazard rate function of the QLD for some parameter values α, θ

2.2 Stress-Strength Reliability for QLD

Assume $X : QLD(\theta_1, \alpha_1)$ and $Y : QLD(\theta_2, \alpha_2)$ are independent random variables with pdf $f(x)$ and $g(x)$, respectively. Then the stress srength reliability can be obtained as:

$R = Pr(Y < X).$

$$= \int_0^\infty \int_0^x f(x)g(y)\ dydx.$$

$$= \int_0^\infty f(x)G(x)\ dx.$$

$$= \int_0^\infty \frac{\theta_1(\alpha_1 + \theta_1 x)}{\alpha_1 + 1} e^{-\theta_1 x} \left[1 - [\frac{(1 + \alpha_2 + \theta_2 x)}{\alpha_2 + 1} e^{-\theta_2 x}] \right] dx.$$

$$= \int_0^\infty \frac{\theta_1(\alpha_1 + \theta_1 x)}{\alpha_1 + 1} e^{-\theta_1 x} dx - \int_0^\infty \frac{\theta_1(\alpha_1 + \theta_1 x)(1 + \alpha_2 + \theta_2 x)\, e^{-(\theta_1 + \theta_2)x}}{(\alpha_1 + 1)(\alpha_2 + 1)} dx.$$

$$= 1 - \frac{\theta_1 \left(2\theta_1\theta_2 + (\theta_1 + \theta_2)(\alpha_2\theta_1 + \alpha_1\theta_2 + \theta_1) + \alpha_1(\alpha_2 + 1)(\theta_1 + \theta_2)^2 \right)}{(\alpha_1 + 1)(\alpha_2 + 1)(\theta_1 + \theta_2)^3}. \qquad (2.5)$$

2.3 Multi-component Stress-Strength Reliability for QLD

Assuming that $F(.)$ and $G(.)$ are quasi Lindley distributions with unknown parameters $\theta_1, \theta_2, (\alpha_1, \alpha_2,$ and that independent random samples $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ are available from $F(.)$ and $G(.)$ respectively. The reliability in multi-component stress-strength for quasi Lindley distribution using Eq.(2.2) is:

$$R_{s,k} = \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty [1 - F(y)]^i [F(y)]^{k-i} dG(y)$$

$$= \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty [1-(1-[\frac{(1+\alpha_1+\theta_1 y)}{\alpha_1+1} e^{-\theta_1 y}])]^i [1-[\frac{1+\alpha_1+\theta_1 y}{\alpha_1+1} e^{-\theta_1 y}]]^{k-i}$$

$$\times [\frac{\theta_2(\alpha_2+\theta_2 y)}{\alpha_2+1}]\, e^{-\theta_2 y} dy.$$

$$= \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty [\frac{1+\alpha_1+\theta_1 y}{\alpha_1+1} e^{-\theta_1 y}]^i \quad [1-[\frac{(1+\alpha_1+\theta_1 y)}{\alpha_1+1} e^{-\theta_1 y}]]^{k-i}$$

$$\times [\frac{\theta_2(\alpha_2+\theta_2 y)}{\alpha_2+1}]\, e^{-\theta_2 y}\, dy.$$

$$= \sum_{i=s}^{k} \sum_{j_1=1}^{k-i} \binom{k}{i} \binom{k-i}{j_1} (-1)^{j_1} \int_0^\infty [\frac{1+\alpha_1+\theta_1 y}{\alpha_1+1}\, e^{-\theta_1 y}]^i [\frac{1+\alpha_1+\theta_1 y}{\alpha_1+1} e^{-\theta_1 y}]^{j_1}$$

$$\times [\frac{\theta_2(\alpha_2+\theta_2 y)}{\alpha_2+1}]\, e^{-\theta_2 y}\, dy.$$

$$= \sum_{i=s}^{k} \sum_{j1=1}^{k-i} \binom{k}{i} \binom{k-i}{j_1} (-1)^{j_1} \int_0^\infty [\frac{(1+\alpha_1)+\theta_1 y}{\alpha_1+1}]^{(i+j_1)} [\frac{\theta_2(\alpha_2+\theta_2 y)}{\alpha_2+1}] e^{(-\theta_2+(i+j_1)\theta_1)y} dy.$$

$$= \sum_{i=s}^{k} \sum_{j_1=1}^{k-i} \binom{k}{i} \binom{k-i}{j_1} (-1)^{j_1} \int_0^\infty [1+\frac{\theta_1 y}{\alpha_1+1}]^{(i+j_1)} [\frac{\theta_2(\alpha_2+\theta_2 y)}{\alpha_2+1}] e^{\,(-\theta_2+(i+j_1)\theta_1)y} dy.$$

$$= \sum_{i=s}^{k} \sum_{j_1=1}^{k-i} \sum_{j_2=1}^{i+j_1} \binom{k}{i} \binom{k-i}{j_1} \binom{i+j_1}{j_2} (-1)^{j_1} \int_0^\infty [\frac{\theta_1}{\alpha_1+1}]^{j_2}\, y^{j_2}\, [\frac{\theta_2(\alpha_2+\theta_2 y)}{\alpha_2+1}]$$

$$\times\, e^{(-\theta_2+(i+j_1)\theta_1)y} dy.$$

$$= \sum_{i=s}^{k} \sum_{j_1=1}^{k-i} \sum_{j_2=1}^{i+j_1} \binom{k}{i} \binom{k-i}{j_1} \binom{i+j_1}{j_2} (-1)^{j_1} [\frac{\theta_1}{\alpha_1+1}]^{j_2} \quad [\frac{\theta_2}{\alpha_2+1}]$$

$$\times \int_0^\infty y^{j_2} (\alpha_2+\theta_2 y) \quad e^{(-\theta_2+(i+j_1)\theta_1)y} dy.$$

$$= \sum_{i=s}^{k} \sum_{j_1=1}^{k-i} \sum_{j_2=1}^{i+j_1} \binom{k}{i} \binom{k-i}{j_1} \binom{i+j_1}{j_2} (-1)^{j_1} [\frac{\theta_2 \theta_1^{\,j_2}}{(\alpha_2+1)(\alpha_1+1)^{j_2}}]$$

$$\times \int_0^\infty [\alpha_2 y^{j_2} e^{(-\theta_2+(i+j_1)\theta_1)y} + \theta_2 y^{j_2+1} e^{(-\theta_2+(i+j_1)\theta_1)y}] dy.$$

$$= \sum_{i=s}^{k}\sum_{j_1=1}^{k-i}\sum_{j_2=1}^{i+j_1}\binom{k}{i}\binom{k-i}{j_1}\binom{i+j_1}{j_2}(-1)^{j_1}\frac{\theta_2\theta_1^{\,j_2}}{(\alpha_2+1)(\alpha_1+1)^{j_2}}$$

$$\times[\alpha_2\frac{\Gamma(j_2+1)}{(\theta_2+(i+j_1)\theta_1)^{j_2+1}}+\theta_2\frac{\Gamma(j_2+2)}{(\theta_2+(i+j_1)\theta_1)^{j_2+2}}].$$

Then,

$$R_{s,k} = \sum_{i=s}^{k}\sum_{j_1=1}^{k-i}\sum_{j_2=1}^{i+j_1}\binom{k}{i}\binom{k-i}{j_1}\binom{i+j_1}{j_2}(-1)^{j_1}\frac{(j_2)!\ \theta_2\theta_1^{\,j_2}}{(\alpha_2+1)(\alpha_1+1)^{j_2}}$$

$$\times[\frac{\alpha_2(\theta_2+(i+j_1)\theta_1)+(j_2+1)\theta_2}{(\theta_2+(i+j_1)\theta_1)^{j_2+2}}]. \qquad (2.6)$$

2.4 Maximum Likelihood Estimation for R

Suppose that $X_1, X_2, \cdots, X_n$ is random sample from $QLD(\theta_1, \alpha_1)$, and $Y_1, Y_2, \cdots, Y_m$ is random sample from $QLD(\theta_2, \alpha_2)$, then the jointly-likelihood function of X and Y is given by:

$$L(x, y; \underline{\phi}) = \prod_{i=0}^{n} f(x_i)\prod_{j=0}^{m} w(y_j).$$

$$= \prod_{i=0}^{n}\frac{\theta_1(\alpha_1+\theta_1 x)}{\alpha_1+1}e^{-\theta_1 x}\prod_{j=0}^{m}\frac{\theta_2(\alpha_2+\theta_2 x)}{\alpha_2+1}e^{-\theta_2 y}.$$

$$= \frac{\theta_1^{\,n}}{(\alpha_1+1)^n}\prod_{i=0}^{n}(\alpha_1+\theta_1 x)\,e^{-\theta_1 x}\times\frac{\theta_2^{\,m}}{(\alpha_2+1)^m}\prod_{j=0}^{m}(\alpha_2+\theta_2 x)e^{-\theta_2 y}.$$

$$= \frac{\theta_1^{\,n}}{(\alpha_1+1)^n}\frac{\theta_2^{\,m}}{(\alpha_2+1)^m}e^{-\sum_{i=1}^{n}\theta_1 x_i}e^{-\sum_{j=1}^{m}\theta_2 y_j}\prod_{i=0}^{n}(\alpha_1+\theta_1 x)\prod_{j=0}^{m}(\alpha_2+\theta_2 x).$$

Then the log likelihood function can be written as:

$$\ell(x,y;\underline{\phi}) = n\log(\theta_1) + m\log(\theta_2) - n\log(\alpha_1+1) - m\log(\alpha_2+1)$$

$$-\theta_1\sum_{i=1}^{n}x_i - \theta_2\sum_{j=1}^{m}y_j + \sum_{i=1}^{n}\log(\alpha_1+\theta_1 x_i) + \sum_{j=1}^{m}\log(\alpha_2+\theta_2 y_j). \tag{2.7}$$

The MLE of $\underline{\phi} = (\theta_1, \alpha_1, \theta_2, \alpha_2)$ can be obtained as a solution of the following equations:

$$\frac{\partial\ell}{\partial\theta_1} = \frac{n}{\theta_1} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{x_i}{\alpha_1+\theta_1 x_i} = 0, \tag{2.8}$$

$$\frac{\partial\ell}{\partial\theta_2} = \frac{m}{\theta_2} - \sum_{j=1}^{m}y_j + \sum_{j=1}^{m}\frac{y_j}{\alpha_2+\theta_2 y_j} = 0, \tag{2.9}$$

$$\frac{\partial\ell}{\partial\alpha_1} = -\frac{n}{\alpha_1+1} + \sum_{i=1}^{n}\frac{1}{\alpha_1+\theta_1 x_i} = 0, \tag{2.10}$$

and

$$\frac{\partial\ell}{\partial\alpha_2} = -\frac{m}{\alpha_2+1} + \sum_{j=1}^{m}\frac{1}{\alpha_2+\theta_2 y_j} = 0. \tag{2.11}$$

Solving these equations numerically using an iterative process as Newton Raphson to get $\hat{\theta}_1, \hat{\alpha}_1, \hat{\theta}_2, \hat{\alpha}_2$. It is well known that the method of maximum likelihood estimation has invariance property, then the MLE of R and $R_{s,k}$ can be obtained as following:

$$\hat{R} = 1 - \frac{\hat{\theta}_1\left(2\hat{\theta}_1\hat{\theta}_2 + (\hat{\theta}_1+\hat{\theta}_2)(\hat{\alpha}_2\hat{\theta}_1 + \hat{\alpha}_1\hat{\theta}_2 + \hat{\theta}_1) + \hat{\alpha}_1(\hat{\alpha}_2+1)(\hat{\theta}_1+\hat{\theta}_2)^2\right)}{(\hat{\alpha}_1+1)(\hat{\alpha}_2+1)(\hat{\theta}_1+\hat{\theta}_2)^3}. \tag{2.12}$$

$$\hat{R}_{s,k}=\sum_{i=s}^{k}\sum_{j_1=1}^{k-i}\sum_{j_2=1}^{i+j_1}\binom{k}{i}\binom{k-i}{j_1}\binom{i+j_1}{j_2}(-1)^{j_1}\frac{(j_2)!\ \ \hat{\theta}_2\hat{\theta}_1^{\,j_2}}{(\hat{\alpha}_2+1)(\hat{\alpha}_1+1)^{j_2}}$$
$$\times[\frac{\hat{\alpha}_2(\hat{\theta}_2+(i+j_1)\hat{\theta}_1)+(j_2+1)\hat{\theta}_2}{(\hat{\theta}_2+(i+j_1)\hat{\theta}_1)^{j_2+2}}]. \qquad (2.13)$$

2.5 Asymptotic Confidence Interval of R

The asymptotic variance-covariance matrix of all parameters can be approximated by the inverse of observed information matrix, and then derive the asymptotic distribution of $\hat{R}$. Based on the asymptotic distribution of $\hat{R}$, we obtain the asymptotic confidence interval of R.

The Fisher information matrix of $\underline{\phi}=(\theta_1,\alpha_1,\theta_2,\alpha_2)$ is given as:

$$I(\underline{\phi})=-\begin{pmatrix} E(\frac{\partial^2 L}{\partial \theta_1^{\,2}}) & E(\frac{\partial^2 L}{\partial \theta_1 \partial \alpha_1}) & E(\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2}) & E(\frac{\partial^2 L}{\partial \theta_1 \partial \alpha_2}) \\ E(\frac{\partial^2 L}{\partial \alpha_1 \partial \theta_1}) & E(\frac{\partial^2 L}{\partial \alpha_1^{\,2}}) & E(\frac{\partial^2 L}{\partial \alpha_1 \partial \theta_2}) & E(\frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2}) \\ E(\frac{\partial^2 L}{\partial \theta_2 \partial \theta_1}) & E(\frac{\partial^2 L}{\partial \theta_2 \partial \alpha_1}) & E(\frac{\partial^2 L}{\partial \theta_2^{\,2}}) & E(\frac{\partial^2 L}{\partial \theta_2 \partial \alpha_2}) \\ E(\frac{\partial^2 L}{\partial \alpha_2 \partial \theta_1}) & E(\frac{\partial^2 L}{\partial \alpha_2 \partial \alpha_1}) & E(\frac{\partial^2 L}{\partial \alpha_2 \partial \theta_2}) & E(\frac{\partial^2 L}{\partial \alpha_2^{\,2}}) \end{pmatrix}.$$

$$=\begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}, \qquad (2.14)$$

where:

$$I_{13} = I_{31} = 0; I_{14} = I_{41} = 0, \tag{2.15}$$

$$I_{23} = I_{32} = 0; I_{24} = I_{42} = 0, \tag{2.16}$$

$$I_{11} = -\frac{n}{\theta_1^2} - \sum_{i=1}^{n} \frac{x_i^2}{(\alpha_1 + \theta_1 x_i)^2}, \tag{2.17}$$

$$I_{22} = \sum_{i=1}^{n} \frac{1}{(\alpha_1 + \theta_1 x_i)^2} - \frac{n}{(\alpha_1 + 1)^2}, \tag{2.18}$$

$$I_{33} = -\frac{m}{\theta_2^2} - \sum_{j=1}^{m} \frac{y_j^2}{(\alpha_2 + \theta_2 y_j)^2}, \tag{2.19}$$

$$I_{44} = \sum_{j=1}^{m} \frac{1}{(\alpha_2 + \theta_2 y_j)^2} - \frac{m}{(\alpha_2 + 1)^2}, \tag{2.20}$$

$$I_{12} = I_{21} = \sum_{i=1}^{n} \frac{x_i}{(\alpha_1 + \theta_1 x_i)^2}, \tag{2.21}$$

and

$$I_{34} = I_{43} = -\sum_{j=1}^{m} \frac{y_j}{(\alpha_2 + \theta_2 y_j)^2}. \tag{2.22}$$

Using the central limit theorem,we obtain the following theorem :

Theorem 2.1: As $n \to \infty$, $m \to \infty$; then

$$(\sqrt{n}(\hat{\theta}_1 - \theta_1), \sqrt{n}(\hat{\alpha}_1 - \alpha_1), \sqrt{m}(\hat{\theta}_2 - \theta_2), \sqrt{m}(\hat{\alpha}_2 - \alpha_2)) \xrightarrow{d} N(0, I^{-1}(\underline{\phi})).$$

Where $\xrightarrow{d}$ means converge in distribution, and $I^{-1}(\underline{\phi})$ is the inverse of the Fisher information matrix $I(\underline{\phi})$.`

In order to establish the asymptotic normality of R, we first define:

$$d(\underline{\phi}) = (\frac{\partial R}{\partial \theta_1}, \frac{\partial R}{\partial \alpha_1}, \frac{\partial R}{\partial \theta_2}, \frac{\partial R}{\partial \alpha_2})^T = (d_1, d_2, d_3, d_4)^T, \tag{2.23}$$

where T is transpose operation, and

$$d_1 = -\frac{\theta_1(2\alpha_1(\alpha_2+1)(\theta_1+\theta_2)+\alpha_1\theta_2+(\alpha_2+1)(\theta_1+\theta_2)+\alpha_2\theta_1+\theta_1+2\theta_2)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3}$$
$$+\frac{3\theta_1\left(\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2+(\theta_1+\theta_2)(\alpha_1\theta_2+\alpha_2\theta_1+\theta_1)+2\theta_1\theta_2\right)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^4}$$
$$-\frac{\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2+(\theta 1+\theta_2)(\alpha_1\theta_2+\alpha_2\theta_1+\theta_1)+2\theta_1\theta_2}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3}, \tag{2.24}$$

$$d_2 = \frac{\theta_1\left(\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2+(\theta_1+\theta_2)(\alpha_1\theta_2+\alpha_2\theta_1+\theta_1)+2\theta_1\theta_2\right)}{(\alpha_1+1)^2(\alpha_2+1)(\theta_1+\theta_2)^3}$$
$$-\frac{\theta_1\left((\alpha_2+1)(\theta_1+\theta_2)^2+\theta_2(\theta_1+\theta_2)\right)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3}, \tag{2.25}$$

$$d_3 = \frac{3\theta_1\left(\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2+(\theta_1+\theta_2)(\alpha_1\theta_2+\alpha_2\theta_1+\theta_1)+2\theta_1\theta_2\right)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^4}$$
$$-\frac{\theta_1(2\alpha_1(\alpha_2+1)(\theta_1+\theta_2)+\alpha_1(\theta_1+\theta_2)+\alpha_1\theta_2+\alpha_2\theta_1+3\theta_1)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3}, \tag{2.26}$$

and

$$d_4 = \frac{\theta_1\left(\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2+(\theta_1+\theta_2)(\alpha_1\theta_2+\alpha_2\theta_1+\theta_1)+2\theta_1\theta_2\right)}{(\alpha_1+1)(\alpha_2+1)^2(\theta_1+\theta_2)^3}$$
$$-\frac{\theta_1\left(\alpha_1(\theta_1+\theta_2)^2+\theta_1(\theta_1+\theta_2)\right)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3}. \tag{2.27}$$

Hence; using **Theorem 2.1**, the asymptotic distribution of $\hat{R}$(the MLE of R) is defined as:

$$\sqrt{n+m}(\hat{R}-R)\overset{d}{\rightarrow}N(0,B), \tag{2.28}$$

where

$$B=Var(\hat{R})=d^{T}(\underline{\phi})I^{-1}(\underline{\phi})d(\underline{\phi}). \tag{2.29}$$

Therefore, using Eq.(2.29), an asymptotic $100(1-\gamma)\%$ confidence interval for R can be obtained as:

$$\hat{R}\pm Z_{\frac{\gamma}{2}}\sqrt{B},$$

where $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ precentile of the standard normal distribution.

2.6 Bayesian Estimation of R

In this section, we provide the Bayes estimate of R where $\theta_1,\theta_2,\alpha_1,\alpha_2$ are unknown parameters and all of these parameters having independent gamma prior distributions as following:

$$\pi(\theta_1)=\frac{b_1^{a_1}}{\Gamma a_1}\theta_1^{a_1-1}e^{-b_1\theta_1},$$

$$\pi(\theta_2)=\frac{b_2^{a_2}}{\Gamma a_2}\theta_2^{a_2-1}e^{-b_2\theta_2},$$

$$\pi(\alpha_1)=\frac{b_3^{a_3}}{\Gamma a_3}\alpha_1^{a_3-1}e^{-b_3\alpha_1},$$

and

$$\pi(\alpha_2)=\frac{b_4^{a_4}}{\Gamma a_4}\alpha_2^{a_4-1}e^{-b_4\alpha_2},$$

The joint posterior pdf is defined as:

$$g(\theta_1,\theta_2,\alpha_1,\alpha_2/data)=\frac{L(x,y/\theta_1,\theta_2,\alpha_1,\alpha_2)\pi(\theta_1)\pi(\theta_2)\pi(\alpha_1)\pi(\alpha_2)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(x,y/\theta_1,\theta_2,\alpha_1,\alpha_2)\pi(\theta_1)\pi(\theta_2)\pi(\alpha_1)\pi(\alpha_2)d\theta_1 d\theta_2 d\alpha_1 d\alpha_2}. \tag{2.30}$$

Then

$$g(\theta_1,\theta_2,\alpha_1,\alpha_2/data)\propto\frac{\theta_1^{\,n}}{(\alpha_1+1)^n}\frac{\theta_2^{\,m}}{(\alpha_2+1)^m}\prod_{i=1}^{n}(\alpha_1+\theta_1 x_i)\prod_{j=1}^{m}(\alpha_2+\theta_2 y_j)\,e^{-\theta_1\sum_{i=1}^{n}x_i}$$
$$\times e^{-\theta_2\sum_{j=1}^{m}y_j}\;\theta_1^{\,a_1-1}e^{-b_1\theta_1}\theta_2^{\,a_2-1}e^{-b_2\theta_2}\alpha_1^{\,a_3-1}e^{-b_3\alpha_1}\alpha_2^{\,a_4-1}e^{-b_4\alpha_2}. \tag{2.31}$$

2.6.1 Bayes estimators under Symmetric and Asymmetric loss functions:

The Bayes estimate of reliability R depending also on the loss function. We discussed before, in the previous chapter, three different loss function; squared error, linex, and general entropy loss functions. The squared error loss function is considered as symmetric loss function, where the linex , and the general entropy loss functions are asymmetric loss functions. In this section we proposed the bayesian estimation of R using these three loss functions such that :

-The Bayes estimate of R under the Se, which is the posterior mean of R, is given by:

$$\hat{R}_{Se}=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R\; g(\theta_1,\theta_2,\alpha_1,\alpha_2/data)d\theta_1 d\theta_2 d\alpha_1 d\alpha_2. \tag{2.32}$$

-The Bayes estimate of R under the Lx, is given by:

$$\hat{R}_{Lx} = \frac{-1}{c} ln[\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-cR} \ g(\theta_1, \theta_2, \alpha_1, \alpha_2 / data) d\theta_1 d\theta_2 d\alpha_1 d\alpha_2], \tag{2.33}$$

where c is constant, $c > 0$, see Zellner (1986).

-The Bayes estimate of R under the Ge, is given by:

$$\hat{R}_{Ge} = [\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R^{-q} \ g(\theta_1, \theta_2, \alpha_1, \alpha_2 / data) d\theta_1 d\theta_2 d\alpha_1 d\alpha_2]^{-1/q}, \tag{2.34}$$

where q is constant, $q > 0$, see Calabria and Pulcini (1994).

These integrals are very complicated to computed analytically, so two different approaches can be used to approximate these integrals, namely, Metropolis-Hastings technique and Importance Sampling technique.

2.6.2 Bayes estimate of R using Metropolis-Hastings technique (MH)

As we mentioned before that MH was developed by Metropolis et al.(1953), and Hastings (1970). The joint posterior density function of $\theta_1, \theta_2, \alpha_1$, and α_2 is given in Eq. (2.31). It is easily seen that the marginal density functions of $\theta_1, \theta_2, \alpha_1$, and α_2 are, respectively:

$$\pi_1(\theta_1 / data) \propto Gamma\left(n + a_1, b_1 + \sum_{i=1}^{n} x_i \right), \tag{2.35}$$

$$\pi_2(\theta_2 / data) \propto Gamma\left(m + a_2, b_2 + \sum_{j=1}^{m} y_j \right), \tag{2.36}$$

$$\pi_3(\alpha_1 / \theta_1, data) \propto \frac{\alpha_1^{a_3 - 1} e^{-b_3 \alpha_1} \prod_{i=1}^{n} (\alpha_1 + \theta_1 x_i)}{(\alpha_1 + 1)^n}, \tag{2.37}$$

and

$$\pi_4(\alpha_2 / \theta_2, data) \propto \frac{\alpha_2^{a_4 - 1} e^{-b_4 \alpha_2} \prod_{j=1}^{m} (\alpha_2 + \theta_2 y_j)}{(\alpha_2 + 1)^m}. \tag{2.38}$$

Therefore, easily samples of θ_1 and θ_2 can be generated by using Gamma distribution as shown in Eqs. (2.35) and (2.36) respectively. However, the posterior distribution of α_1, α_2 cannot be generated from a well known distributions. The MH algorithm, can be used to solve this problem, as shown in the following algorithm.

- Step1: Start with initial value of α_1, α_2 such that $\alpha_1^{(0)} = \hat{\alpha}_1$, and

  $\alpha_2^{(0)} = \hat{\alpha}_2$.

- Step2: Set $i = 1$.

- Step3: Generate $\theta_1^{(i)}$ from $\pi_1(\theta_1/data)$.

- Step4: Generate $\theta_2^{(i)}$ from $\pi_2(\theta_2/data)$.

- Step5: Generate $\alpha_1^{(i)}$ from $\pi_3(\alpha_1/\theta_1, data)$ using the MH

algorithm with the proposal distribution q_1 as following:

- Generate $\alpha_1^{(*)}$ from the proposal distribution

$$q_1 = N(\alpha_1^{(i-1)}, Var(\alpha_1^{(i-1)})).$$

- Calculate the acceptance probability

$$r_1(\alpha_1^{(i-1)}, \alpha_1^{(*)}) = Min[1, \frac{\pi_3(\alpha_1^{(*)} \mid \theta_1^{(i)}, data)}{\pi_3(\alpha_1^{(i-1)} \mid \theta_1^{(i)}, data)}].$$

-Generate U from $U(0,1)$.

- If $U \le r_1(\alpha_1^{(i-1)}, \alpha_1^{(*)})$, accept the proposal distribution and set

$$\alpha_1^{(i)} = \alpha_1^{(*)} \text{, otherwise set } \alpha_1^{(i)} = \alpha_1^{(i-1)}.$$

- Step6: Generate $\alpha_2^{(i)}$ from $\pi_4(\alpha_2/\theta_2, data)$ using the MH

algorithm with the proposal distribution q_2 as following:

- Generate $\alpha_2^{(*)}$ from the proposal distribution

$$q_2 = N(\alpha_2^{(i-1)}, Var(\alpha_2^{(i-1)})).$$

- Calculate the acceptance probability

$$r_2(\alpha_2^{(i-1)}, \alpha_2^{(*)}) = Min[1, \frac{\pi_4(\alpha_2^{(*)} \mid \theta_2^{(i)}, data)}{\pi_4(\alpha_2^{(i-1)} \mid \theta_2^{(i)}, data)}].$$

- Generate U from $U(0,1)$.

- If $U \le r_2(\alpha_2^{(i-1)}, \alpha_2^{(*)})$, accept the proposal distribution and set

$\alpha_2^{(i)} = \alpha_2^{(*)}$, otherwise set $\alpha_2^{(i)} = \alpha_2^{(i-1)}$.

- Step7: Compute $R^{(i)}$ at $(\theta_1^{(i)}, \theta_2^{(i)}, \alpha_1^{(i)}, \alpha_2^{(i)})$ using Eq.(2.5).
- Step8: Set $i = i + 1$.
- Step9: Repeat steps from $(3-8)$ N times.

Then;

-An approximate Bayes estimate of R under Seloss function is given as:

$$\hat{R}MH_{Se} = \frac{1}{N-M} \sum_{i=M+1}^{N} R^{(i)}. \tag{2.39}$$

-An approximate Bayes estimate of R under Lx loss function is given as:

$$\hat{R}MH_{Lx} = \frac{-1}{c} \log\left[\frac{1}{N-M} \sum_{i=M+1}^{N} e^{-cR^{(i)}}\right]. \tag{2.40}$$

-An approximate Bayes estimate of R under Ge loss function is given as:

$$\hat{R}MH_{Ge} = \left[\frac{1}{N-M} \sum_{i=M+1}^{N} (R^{(i)})^{-q}\right]^{-1/q}, \tag{2.41}$$

where M is the burn-in samples, N is the MCMC samples.

2.6.3 Bayes estimate of R using Importance Sampling technique (IS)

Importance Sampling Technique has suggested by Chen and Shao(1999). In statistics, importance sampling is the name for the general technique of determining the properties of a distribution by drawing samples from another distribution. The focus of importance sampling here is to determine as easily and accurately as possible the properties of the posterior from a representative sample from the second distribution.

Using Importance Sampling Technique, Eq.(2.31) can be written as

$$g(\theta_1,\theta_2,\alpha_1,\alpha_2/data) \propto g_1(\theta_1/data)g_2(\theta_2/data)g_3(\alpha_1/data) \times g_4(\alpha_2/data)h(\theta_1,\theta_2,\alpha_1,\alpha_2/data). \quad (2.42)$$

where:

$$g_1(\theta_1/data) \propto Gamma(n+a_1,b_1+\sum_{i=1}^{n}x_i), (2.43)$$

$$g_2(\theta_2/data) \propto Gamma(m+a_2,b_2+\sum_{j=1}^{m}y_j), \quad (2.44)$$

$$g_3(\alpha_1/data) \propto Gamma(a_3,b_3), \quad (2.45)$$

$$g_4(\alpha_2/data) \propto Gamma(a_4,b_4), \quad (2.46)$$

and

$$h(\theta_1,\theta_2,\alpha_1,\alpha_2/data) = \frac{\prod_{i=1}^{n}(\alpha_1+\theta_1 x_i) \quad \prod_{j=1}^{m}(\alpha_2+\theta_2 y_j)}{(\alpha_1+1)^n \quad (\alpha_2+1)^m}. \quad (2.47)$$

As shown, all the above functions from $g_1(\theta_1/data)$ to $g_4(\alpha_2/data)$ follow gamma distributions with different parameters, so it is quite simple to generate QLD parameters from them. Assuming that $a_1,a_2,\cdots,a_4$ and $b_1,b_2,\cdots,b_4$ are known, and assuming initial values for $\theta_1,\theta_2,\alpha_1,\alpha_2$. we can use the following Importance Sampling Algorithm:

- Step1: Generate θ_{11} from $g_1(\theta_1/data)$.
- Step2: Generate θ_{21} from $g_2(\theta_2/data)$.

• Step3: Generate α_{11} from $g_3(\alpha_1/data)$.

• Step4: Generate α_{21} from $g_4(\alpha_2/data)$.

• Step5: Repeat steps from 1 to 4, N times to obtain

$$(\theta_{11},\theta_{21},\alpha_{11},\alpha_{21}),\cdots,(\theta_{1N},\theta_{2N},\alpha_{1N},\alpha_{2N}).$$

Then

-An approximate Bayes estimate of R under Se loss function can be obtained as

$$\hat{R}IS_{Se} = \frac{\sum_{i=1}^{N} R_i\, h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}{\sum_{i=1}^{N} h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}. \tag{2.48}$$

- An approximate Bayes estimate of R under Lxloss function can be obtained as

$$\hat{R}IS_{Lx} = \frac{-1}{c}\log[\frac{\sum_{i=1}^{N} e^{-cR_i}\ h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}{\sum_{i=1}^{N} h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}]. \tag{2.49}$$

-An approximate Bayes estimate of R under Ge loss function can be obtained as:

$$\hat{R}IS_{Ge} = [\frac{\sum_{i=1}^{N} R_i^{-q}\ h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}{\sum_{i=1}^{N} h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}]^{-1/q}, \tag{2.50}$$

where

$R_i = R(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i})$. as defined in Eq.(2.5), for $i = 1,\cdots,N$.

2.7 Numerical Study

In this section, we mainly present some simulation experiments to see the performance of the mentioned methods for different sample sizes, $(n,m) = (10,10), (20,20), (30,30), (50,50), (70,70), (100,100)$. We simulated 1000 complete samples from quasi Lindley distribution with the parameter values; $\theta_1 = 0.2, \theta_2 = 1.5, \alpha_1 = 2, \alpha_2 = 0.8,$ with true reliability value is 0.87399 .

We also compute the 95% confidence intervals of R based on the observed Fisher information matrix. We compared the performances of the MLE and the Bayes estimates in terms of mean squared errors (MSE's). Also two different techniques of Bayesian estimation (IS, MH) are compared for different loss error functions. Bayesian estimation for different loss error functions was proposed with different values of c, q such that;

$$c_1 = -3 \text{ (Lx1)},\ c_2 = 5 \text{ (Lx2)},\ q_1 = -3 \text{ (Ge1)},\ q_2 = 5 \text{ (Ge2)}.$$

Bayesian estimation studied under the informative gamma priors. For choosing suitable hyper-parameters, the experimenters can incorporate their prior guess in terms of location and precision for the parameter of interest. The gamma distribution for the priors has $mean = a/b$, and $variance = a/b^2$. We assume a small value of prior variance (0.01), and take the mean equal to the true value of the parameter of interest. For each parameter prior we solve the two equations of the mean and the variance, we obtain the following values of hyper-parameters :

$$a_1 = 4, a_2 = 225, a_3 = 400, a_4 = 64, \text{ and}$$
$$b_1 = 20, b_2 = 150, b_3 = 200, b_4 = 80.$$

We also computed the Bayes estimates based on 11000 samples and discard the first 1000 values as burn-in.The maximum likelihood estimator and asymptotic confidence intervals of R for different (n,m) are obtained in

Table 2.1. Bayes estimates of R using different techniques under different loss error functions are obtained in Table 2.2.

Table 2.1: Average estimate, (MSEs) for MLE, and average confidence length of the simulated 95% confidence intervals of R. (all MSE values are multiplied by 10^{-3})

Sample size	Average estimate (MSE)	C.I.L	C.I.U	C.I.length
(10,10)	0.8661 (3.2129)	0.734952	0.997237	0.262
(20,20)	0.8782 (1.1262)	0.7872	0.9692	0.182
(30,30)	0.8748 (0.7609)	0.7976	0.9521	0.155
(50,50)	0.8758 (0.459333)	0.8167	0.9371	0.120
(70,70)	0.8752 (0.3349)	0.8237	0.9266	0.103
(100,100)	0.8774 (0.1554)	0.8349	0.9199	0.084

Table 2.2: Average estimates (mean squared error) of R for different bayes estimators under different error loss functions. (all MSE values are multiplied by 10^{-3})

Est.	Importance Sampling					MH Technique				
(n,m)	Se	Lx1	Lx2	Ge1	Ge2	Se	Lx1	Lx2	Ge1	Ge2
(10,10)	0.8671 (0.7516)	0.8686 (0.7058)	0.8646 (0.8421)	0.8683 (0.7134)	0.8635 (0.8934)	0.8879 (0.7611)	0.8891 (0.7748)	0.8858 (0.7459)	0.8888 (0.77)	0.8849 (0.7477)
(20,20)	0.8706 (0.4769)	0.8715 (0.4611)	0.8692 (0.5075)	0.8713 (0.4637)	0.8686 (0.5239)	0.8919 (0.6747)	0.8927 (0.692)	0.8908 (0.6479)	0.8925 (0.6872)	0.8904 (0.6403)
(30,30)	0.8707 (0.3591)	0.8714 (0.3491)	0.8697 (0.3784)	0.8712 (0.3508)	0.8693 (0.3883)	0.8916 (0.5684)	0.8921 (0.5822)	0.8907 (0.5464)	0.8919 (0.5785)	0.8904 (0.5396)
(50,50)	0.8717 (0.2536)	0.8721 (0.24863)	0.8711 (0.2626)	0.8719 (0.24957)	0.8708 (0.2670)	0.8899 (0.4366)	0.8903 (0.4458)	0.8894 (0.4216)	0.8902 (0.4434)	0.8892 (0.4167)
(70,70)	0.8734 (0.2189	0.8737 (0.217)	0.8729 (0.2226)	0.8737 (0.2173)	0.8729 (0.2244)	0.8887 (0.3755)	0.8890 (0.3823)	0.8883 (0.3645)	0.8888 (0.3805)	0.8881 (0.3609)
(100,100)	0.87525 (0.1384)	0.8754 (0.1387)	0.8749 (0.138)	0.8754 (0.1386)	0.8749 (0.138)	0.8859 (0.2432)	0.8862 (0.2477)	0.8856 (0.2359)	0.8861 (0.2465)	0.8854 (0.2334)

2.8 Real Data Analysis

In this section we present the analysis of real data, introduced by Singh et al. (2014). The data represent the waiting times (in minutes) before customer service of two banks A and B, respectively. The use of Lindley distribution for the waiting times (bank A) data has been originally discussed by Lindley (1958). Since then, many authors have suggested the data under different set-up for Lindley distribution. We are interested in estimating the stress-strength parameter R = P(Y < X) where X and Y denotes the customer service time in Bank A and B (Data set 1, 2) respectively. The data sets are presented below:

Data set 1: $\mathbf{X}(n=100)$

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2,3.3, 3.5, 3.6,4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

Data set 2: $\mathbf{Y}(m=60)$

0.1, 0.2, 0.3, 0.7, 0.9, 1.1, 1.2, 1.8, 1.9, 2.0, 2.2, 2.3, 2.3, 2.3, 2.5, 2.6, 2.7, 2.7, 2.9, 3.1, 3.1, 3.2, 3.4, 3.4, 3.5, 3.9, 4.0, 4.2, 4.5, 4.7, 5.3, 5.6, 5.6, 6.2, 6.3, 6.6, 6.8, 7.3, 7.5, 7.7, 7.7, 8.0, 8.0, 8.5, 8.5, 8.7, 9.5, 10.7, 10.9, 11.0, 12.1, 12.3, 12.8, 12.9, 13.2, 13.7, 14.5, 16.0, 16.5, 28.0.

First,we checked the suitability of the considered real data setsto the QLD.Therefore we distribution. The fitting summary has been presented in Table 2.3, which indicates that the QLD fits well to data Set 1 and data Set 2.

Table 2.3: P-value of different goodness-of-fit tests for data set 1, 2.

Test	K-S	A-D	C-V
data set 1.	0.0654	0.0217	0.0501
data set 2.	0.9287	0.965	0.9310

Based on the MLEs $\hat{\theta}_1, \hat{\alpha}_1, \hat{\theta}_2, \hat{\alpha}_2$ the point estimate of R is 0.59 and the 95% confidence interval of R is (0.25, 0.93). For real data sets, the maximum likelihood and Bayes estimates of the stress-strength parameters and reliability are summarized in Table 2.4.

Table 2.4: The MLEs and Bayes estimates of stress-strength parameters and reliability R from real data sets

Est.	θ_1	θ_1	α_1	α_2	R
MLE	0.1	0.27	84.09	0.41	0.59
$Bayes_{IS}$	0.13	0.59	1.44	0.66	0.81
$Bayes_{MH}$	0.1	0.54	22.13	0.69	0.77

2.9 Conclusions

In this chapter, maximum likelihood and Bayesian estimation methods for stress-strength reliability R were discussed, when X and Y both follow a QLD with different parameters. We obtained the 95% confidence intervals of R based on the observed Fisher information matrix. We proposed the Bayesian estimation based on independent gamma priors under different error loss functions(Se, Lx, and Ge). we suggested the IS and MH techniques to generate samples from the posterior distributions and then compute the Bayes estimates. Simulation study has been introduced to investigate the performance and compare among all mentioned methods.

Therefore, from the results presented earlier in Tables 2.1 and 2.2 , we observed that:

- The performance of the Bayes estimators is better than maximum likelihood for all different sample sizes.
- Mean squared error(MSE's) for all estimation methods decrease as sample size increase.
- As sample size increased, the asymptotic confidence intervals for R are

improving, and thier length decreased. That means the estimated reliability becomes in the most accurate interval.

• Maximum likelihood results are improving and become closer to Bayesian results as sample size increased.

• For Bayes estimators, IS technique gives less MSE's values, so it is better than MH technique for the same priors values, and the same number of generated samples.

• Ge, and Lx loss functions gave less MSE's at specified values of c, q. As shown above Lx2,Ge2 acheived the best results for MH, but for IS technique Lx1,Ge1 are the best estimators .

A real data analysis has been performed for illustrative purposes.

CHAPTER 3

Estimation of Stress-Strength Reliability for Exponentiated Generalized Inverse Weibull Distribution

3.1 Introduction

One of the most widely used lifetime distributions in reliability analysis is the Inverse Weibull distribution (IW). It can be used to determine the maintenance periods of reliability centered maintenance activities. It can also be used to model a variety of failure characteristics such as infant mortality, useful life and wear- out periods and applications in medicine, reliability and ecology. Keller et al.(1982) discussed the use of the IW distribution as a suitable model to describe the degeneration phenomena of mechanical components such as the dynamic components (pistons, crankshaft, etc.) of diesel engines. Nelson (1982) provided a good fit to several data such as the times to breakdown of an insulating fluid using the IW distribution, subject to the action of constant tension. Calabria and Pulcini (1994) suggested the IW distribution for Bayes 2-sample prediction.

The IW distribution has a cumulative distribution function (CDF):

$$V(x) = e^{-(\frac{\lambda}{x})^{\theta}}, x > 0, \lambda, \theta > 0, \tag{3.1}$$

and a probability density function (pdf):

$$v(x) = \theta\lambda^{\theta} x^{-\theta-1} e^{-(\frac{\lambda}{x})^{\theta}}. \tag{3.2}$$

Researchers always seeking for developing new and more flexible distributions. As a result, many new distributions of (IW) have been developed and studied. Cordeiro et al. (2013) proposed a new class of distributions that extend the exponentiated type distributions and they obtained some of its structural properties. Given a continuous CDF V(x) , they defined the

Exponentiated Generalized (EG) class of distributions by:

$$F(x) = [1-(1-V(x))^{\alpha}]^{\beta}, \tag{3.3}$$

where $\alpha > 0$ and $\beta > 0$ are two additional shape parameters. The probability density function (pdf) of this new class has the form:

$$f(x) = \alpha\beta v(x)[1-V(x)]^{\alpha-1}[1-(1-V(x))^{\alpha}]^{\beta-1}, \tag{3.4}$$

This class of distributions extends a various exponentiated type distributions. The EG family of densities (3.4) allows for greater exibility of its tails and can be widely applied in many areas of engineering and biology. Note that even if v(x) is a symmetric distribution, the distribution f(x) will not be a symmetric distribution. The two extra parameters α, β can control the new distribution properties.

This chapter discussed the stress-strength reliability model $R = Pr(Y < X)$ when X and Y have an Exponentiated Generalized Inverse Weibull (EGIW) distribution with different parameters. The problem of stress-strength reliability is studied to obtain the reliability function of the parameters of EGIW distribution. Reliability for multi-component stress-strength model for EGIW distribution is also studied. Maximum likelihood estimation for stress-strength reliability of underlying distribution is performed. Bayesian estimator of R is obtained using importance sampling technique. A simulation study to investigate and compare the performance of each method of estimation is performed. Finally analysis of a real data set has also been presented for illustrative purposes.

3.2 The Exponentiated Generalized Inverse Weibull Distribution (EGIW)

The Exponentiated Generalized Inverse Weibull Distribution (EGIW) was introduced by Elbatal and Muhammed(2014) as extension of exponentiated generalized family. They had provided a comprehensive study for this distribution, they derive the moment generating function and the r^{th} moment thus generalizing some results in the literature. Also , Expressions for the density, moment generating function and r^{th} moment of the order statistics are obtained in their paper.

The EGIW distribution has a pdf f(x) and CDF F(x):

$$f(x)=\alpha\beta\theta\lambda^{\theta}x^{-\theta-1}e^{-(\frac{\lambda}{x})^{\theta}}[1-e^{-(\frac{\lambda}{x})^{\theta}}]^{\alpha-1}[1-(1-e^{-(\frac{\lambda}{x})^{\theta}})^{\alpha}]^{\beta-1}, \qquad (3.5)$$

$$F(x)=[1-(1-e^{-(\frac{\lambda}{x})^{\theta}})^{\alpha}]^{\beta}, \qquad (3.6)$$

where

$$x>0,\lambda,\theta,\alpha,\beta>0.$$

Figures 3.1 and 3.2 illustrate pdf and CDF of (EGIW) distribution for selected values of the parameters.

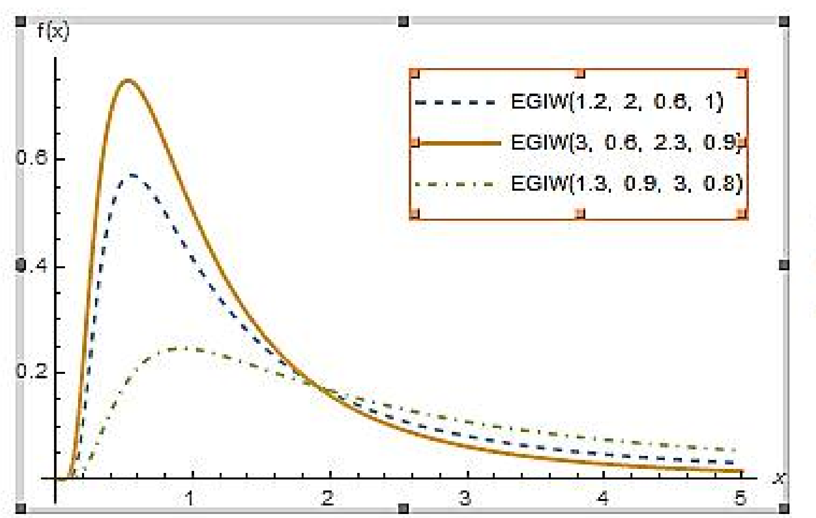

Figure 3.1: Pdf of the EGIW distribution for some parameter values $\alpha,\beta,\lambda,$ and θ

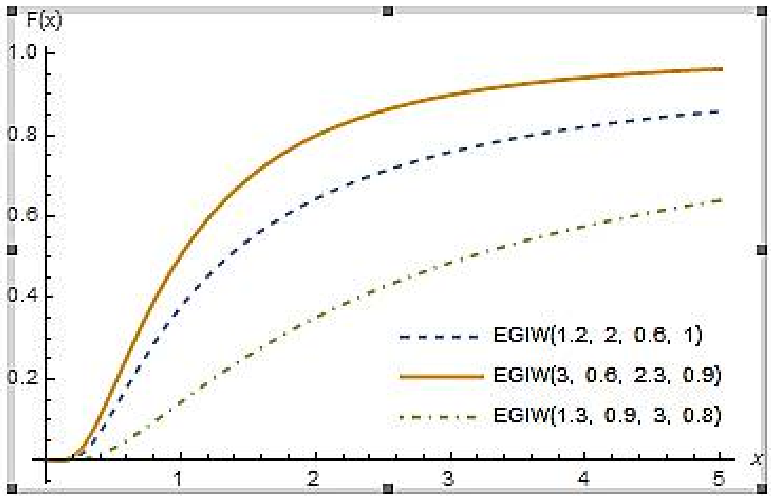

Figure 3.2: CDF of the EGIW distribution for some parameter values $\alpha, \beta, \lambda,$ and θ

The survival and hazard (failure) rate functions of the (EGIW) distribution are given respectively by:

$$S(x) = 1 - [1 - (1 - e^{-(\frac{\lambda}{x})^{\theta}})^{\alpha}]^{\beta}, \qquad (3.7)$$

and

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\alpha\beta\theta\lambda^{\theta} x^{-\theta-1} e^{-(\frac{\lambda}{x})^{\theta}} [1 - e^{-(\frac{\lambda}{x})^{\theta}}]^{\alpha-1} [1 - (1 - e^{-(\frac{\lambda}{x})^{\theta}})^{\alpha}]^{\beta-1}}{1 - [1 - (1 - e^{-(\frac{\lambda}{x})^{\theta}})^{\alpha}]^{\beta}} \qquad (3.8)$$

Figures 3.3 and 3.4 illustrate survival and hazard (failure) rate functions of EGIW distribution for selected values of the parameters.

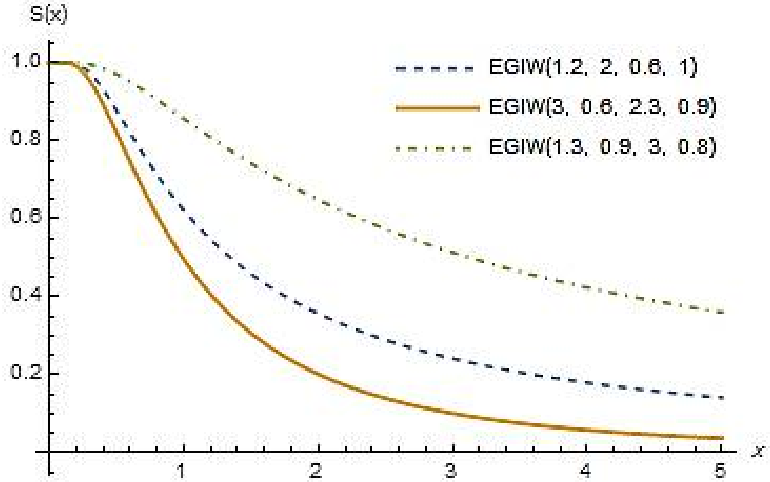

Figure 3.3: The survival function of the EGIW distribution for some parameter values $\alpha, \beta, \lambda,$ and θ

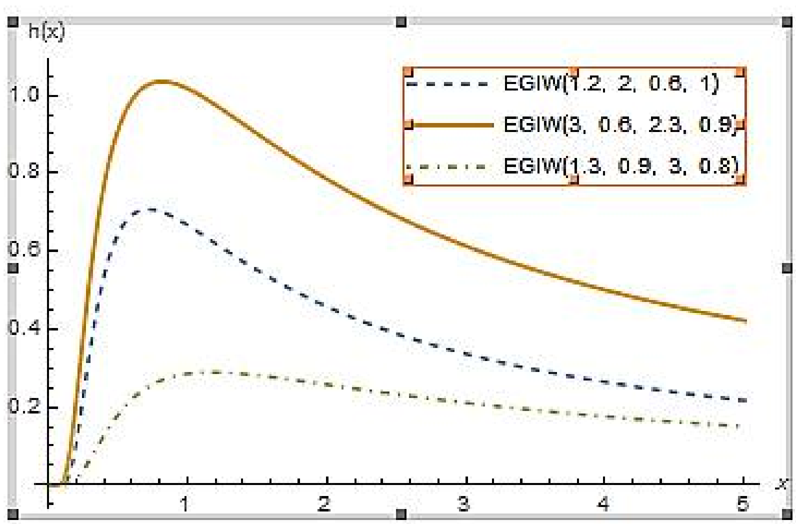

Figure 3.4: The hazard rate function of the EGIW distribution for some parameter values $\alpha, \beta, \lambda,$ and θ

The EGIW distribution is very flexible model that approaches to different distributions when its parameters are changed. Its flexibility is explained in the following, if X is a random variable with pdf in Eq.(3.5), then we have the following special cases:

- If $\alpha = \beta = 1$, then Eq.(3.5) reduces to the inverse weibull distribution.
- If $\alpha = 1$, then we get the exponentiated (generalized) inverse weibull distribution.
- If $\theta = 1$, then we get the exponentiated generalized inverse exponential distribution.
- If $\alpha = \beta = \theta = 1$, then we get the inverse exponential distribution.

3.3Expansions for The ProbabilityDensity and Cumulative Distribution Functions

In this section, we present a new representations for the pdf and the CDF of (EGIW). Equations (3.5) and (3.6) are straightforward to compute using any software with algebraic facilities, but the integration for get the reliability will be very difficult using these formula. So the mathematical relation given below will be useful in next sections.

If b is a positive real non integer and $|z| \leq 1$, we have the power series expansion

$$[1-z]^{b-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} z^j,$$

where

$$\binom{b-1}{j} = \frac{\Gamma(b)}{j!\,\Gamma(b-j)}.$$

Applying this in Eqs.(3.5) and (3.6), and using fractional binomial theorem (See Chan et al. (2006)), we have:

$$f(x) = \alpha\beta g(x)[1-G(x)]^{\alpha-1}[1-(1-G(x))^{\alpha}]^{\beta-1}$$

$$= \alpha\beta g(x)[1-G(x)]^{\alpha-1} \sum_{j_1=0}^{\infty} (-1)^{j_1} \binom{\beta-1}{j_1} [1-G(x)]^{\alpha j_1}$$

$$= \alpha\beta g(x) \sum_{j_1=0}^{\infty} (-1)^{j_1} \binom{\beta-1}{j_1} [1-G(x)]^{\alpha(j_1+1)-1}$$

$$= \alpha\beta g(x) \sum_{j_1=0}^{\infty} (-1)^{j_1} \binom{\beta-1}{j_1} \sum_{j_2=0}^{\infty} (-1)^{j_2} \binom{\alpha(j_1+1)-1}{j_2} [G(x)]^{j_2}$$

$$= \alpha\beta \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} (-1)^{(j_1+j_2)} \binom{\beta-1}{j_1} \binom{\alpha(j_1+1)-1}{j_2} g(x)[G(x)]^{j_2}$$

$$= \alpha\beta \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} (-1)^{(j_1+j_2)} \binom{\beta-1}{j_1} \binom{\alpha(j_1+1)-1}{j_2} \theta\lambda^{\theta} x^{-\theta-1} e^{-(\frac{\lambda}{x})^{\theta}} [e^{-(\frac{\lambda}{x})^{\theta}}]^{j_2}$$

$$= \alpha\beta\theta\lambda^{\theta} x^{-\theta-1} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} (-1)^{(j_1+j_2)} \binom{\beta-1}{j_1} \binom{\alpha(j_1+1)-1}{j_2} \exp^{-(j_2+1)(\frac{\lambda}{x})^{\theta}} .$$

(3.9)

Likelly for F(x),

$$F(x) = [1-(1-G(X))^{\alpha}]^{\beta}$$

$$= \sum_{j_3=0}^{\infty} (-1)^{j_3} \binom{\beta}{j_3} [1-G(x)]^{\alpha j_3}$$

$$= \sum_{j_3=0}^{\infty} (-1)^{j_3} \binom{\beta}{j_3} \sum_{j_4=0}^{\infty} (-1)^{j_4} \binom{\alpha j_3}{j_4} [G(x)]^{j_4}$$

$$= \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} (-1)^{(j_3+j_4)} \binom{\beta}{j_3} \binom{\alpha j_3}{j_4} [G(x)]^{j_4}$$

$$= \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} (-1)^{(j_3+j_4)} \binom{\beta}{j_3} \binom{\alpha j_3}{j_4} \exp^{-j_4(\frac{\lambda}{x})^{\theta}} ,$$

(3.10)

where $g(x) = \theta\lambda^{\theta} x^{-\theta-1} e^{-(\frac{\lambda}{x})^{\theta}}$, $G(x) = e^{-(\frac{\lambda}{x})^{\theta}}$.

3.4 Stress-Strength Reliability Form

Suppose we have two independent random variables X (represents the strength of some component) and Y (represents the stress applied to the component), with pdf $f(x)$ and $w(y)$, respectively. Let $X : EGIW(\alpha_1, \beta_1, \lambda, \theta)$ and $Y : EGIW(\alpha_2, \beta_2, \lambda, \theta)$.

Then, the stress-strength reliability function is given by:

$$R = Pr(Y < X)$$

$$= \int_0^{\infty}\int_0^{x} f(x)w(y)\ \ dydx.$$

$$= \int_0^{\infty} f(x)W_x(y)\ \ dx \tag{3.11}$$

Using Eqs.(3.9) and (3.10) we have the following:

$$R = \int_0^{\infty} \alpha_1\beta_1\theta\lambda^{\theta}x^{-\theta-1}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}(-1)^{(j_1+j_2+j_3+j_4)}\binom{\beta-1}{j_1}\binom{\alpha(j_1+1)-1}{j_2}\ e^{-(j_2+1)(\frac{\lambda}{x})^{\theta}}$$

$$\times\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{\infty}\beta_2\binom{\beta_2}{j_3}\binom{\alpha_2 j_3}{j_4}\ e^{-j_4(\frac{\lambda}{x})^{\theta}}\ dx.$$

$$= \alpha_1\beta_1\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{\infty}(-1)^{(j_1+j_2+j_3+j_4)}\binom{\beta-1}{j_1}\binom{\alpha(j_1+1)-1}{j_2}\binom{\beta_2}{j_3}\binom{\alpha_2 j_3}{j_4}$$

$$\times\int_0^{\infty}\theta\lambda^{\theta}x^{-\theta-1}e^{-(j_2+j_4+1)(\frac{\lambda}{x})^{\theta}}\ dx.$$

$$= \alpha_1\beta_1\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{\infty}(-1)^{(j_1+j_2+j_3+j_4)}\binom{\beta-1}{j_1}\binom{\alpha(j_1+1)-1}{j_2}$$

$$\times\binom{\beta_2}{j_3}\binom{\alpha_2 j_3}{j_4}\frac{1}{(j_2+j_4+1)}. \tag{3.12}$$

Note that when the exponents in Eqs. (3.5) and (3.6) are integers, the expansions in Eqs.(3.9), (3.10) and (3.12) become finite and this is a special case from fractional binomial theorem.

3.5 Reliability for Multi-Component Stress-Strength Model

The reliability for a multi-component stress-strength model has developed by Bhattacharya and Johnson (1974). Let the random samples Y, $X_1, X_2, \cdots, X_k$ being independent, $G(y)$ be the continuous distribution function of Y and $F(x)$ be the common continuous distribution function of $X_1, X_2, \cdots, X_k$. Suppose that a system, with k identical components functions if s, $(1 \le s \le k)$, or more of the components simultaneously operate.

In this operating environment, the system is subjected to a stress Y which is a random variable with distribution function $G(.)$.The strengths of the components, that is the minimum stress to cause failure, are independent and identically distributed random variables with distribution function $F(.)$. The system reliability, which is the probability that the system does not fail, is the function $R_{s,k}$ given by:

$$\begin{aligned} R_{s,k} &= Pr[\text{at least } s \text{ of the } (X_1, X_2, \cdots, X_k) \text{ exceed } Y] \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty [1 - F(y)]^i [F(y)]^{k-i} \, dG(y). \end{aligned}$$

(3.13)

From Eqs. (3.5) and (3.6), The reliability for multi-component stress-strength of EGIW distribution is:

$$R_{s,k} = \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty \left[1 - \left(1 - (1 - e^{-(\frac{\lambda}{y})^\theta})^{\alpha_1} \right)^{\beta_1} \right]^i \left[\left(1 - (1 - e^{-(\frac{\lambda}{y})^\theta})^{\alpha_1} \right)^{\beta_1} \right]^{k-i}$$

$$\times \alpha_2 \beta_2 \theta \lambda^\theta y^{-\theta-1} e^{-(\frac{\lambda}{y})^\theta} \left[1 - e^{-(\frac{\lambda}{y})^\theta} \right]^{\alpha_2 - 1} \left[1 - (1 - e^{-(\frac{\lambda}{y})^\theta})^{\alpha_2} \right]^{\beta_2 - 1} dy.$$

(3.14)

Assume that $t=(1-e^{-(\frac{\lambda}{y})^{\theta}})$, we obtain:

$$R_{s,k}=\sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left[1-\left(1-t^{\alpha_1}\right)^{\beta_1}\right]^i\left[1-t^{\alpha_1}\right]^{\beta_1(k-i)}$$

$$\times\alpha_2\beta_2\theta\lambda^{\theta}y^{-\theta-1}e^{-(\frac{\lambda}{y})^{\theta}}t^{\alpha_2-1}\left[1-t^{\alpha_2}\right]^{\beta_2-1}dy.$$

$$=\sum_{i=s}^{k}\binom{k}{i}\sum_{j_1=0}^{i}\binom{i}{j_1}(-1)^{j_1}\int_0^{\infty}\left[1-t^{\alpha_1}\right]^{\beta_1(j_1+k-i)}$$

$$\times\alpha_2\beta_2\theta\lambda^{\theta}y^{-\theta-1}e^{-(\frac{\lambda}{y})^{\theta}}t^{\alpha_2-1}\left[1-t^{\alpha_2}\right]^{\beta_2-1}dy.$$

$$=\sum_{i=s}^{k}\sum_{j_1=0}^{i}\sum_{j_2=0}^{\beta_1(j_1+k-i)}\sum_{j_3=0}^{\beta_2-1}\binom{k}{i}\binom{i}{j_1}\binom{\beta_1(j_1+k-i)}{j_2}\binom{\beta_2-1}{j_3}(-1)^{j_1+j_2+j_3}\alpha_2\beta_2$$

$$\times\int_0^{\infty}\theta\lambda^{\theta}y^{-\theta-1}e^{-(\frac{\lambda}{y})^{\theta}}t^{\alpha_1 j_2+\alpha_2 j_3+\alpha_2-1}dy.$$

$$=\sum_{i=s}^{k}\sum_{j_1=0}^{i}\sum_{j_2=0}^{\beta_1(j_1+k-i)}\sum_{j_3=0}^{\beta_2-1}\binom{k}{i}\binom{i}{j_1}\binom{\beta_1(j_1+k-i)}{j_2}\binom{\beta_2-1}{j_3}(-1)^{j_1+j_2+j_3}\alpha_2\beta_2$$

$$\times\int_0^{1}t^{\alpha_1 j_2+\alpha_2(j_3+1)-1}dt$$

$$=\sum_{i=s}^{k}\sum_{j_1=0}^{i}\sum_{j_2=0}^{\beta_1(j_1+k-i)}\sum_{j_3=0}^{\beta_2-1}\binom{k}{i}\binom{i}{j_1}\binom{\beta_1(j_1+k-i)}{j_2}\binom{\beta_2-1}{j_3}$$

$$\times(-1)^{j_1+j_2+j_3}\frac{\alpha_2\beta_2}{\alpha_1 j_2+\alpha_2(j_3+1)}.$$

(3.15)

3.6 Maximum Likelihood Estimation for R

In this section, the maximum likelihood estimator (MLE) of R is derived. Suppose $X_1,X_2,\cdots,X_n$ is random sample from $EGIW(\alpha_1,\beta_1,\lambda,\theta)$, and $Y_1,Y_2,\cdots,Y_m$ is random sample from $EGIW(\alpha_2,\beta_2,\lambda,\theta)$. Now, to get the MLE of R we first get the maximum likelihood estimates for the parameters of X and Y; $\underline{\phi} = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda, \theta)$. Since X, Y are independent random variables, then the jointly-likelihood function of X and Y is:

$$L(x, y; \underline{\phi}) = \prod_{i=0}^{n} f(x_i) \prod_{j=0}^{m} w(y_j).$$

$$= \prod_{i=0}^{n} \alpha_1 \beta_1 \theta \lambda^{\theta} x_i^{-\theta-1} e^{-(\frac{\lambda}{x_i})^{\theta}} [1 - e^{-(\frac{\lambda}{x_i})^{\theta}}]^{\alpha_1 - 1} [1 - (1 - e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}]^{\beta_1 - 1}$$

$$\times \prod_{j=0}^{m} \alpha_2 \beta_2 \theta \lambda^{\theta} y_j^{-\theta-1} e^{-(\frac{\lambda}{y_j})^{\theta}} [1 - e^{-(\frac{\lambda}{y_j})^{\theta}}]^{\alpha_2 - 1} [1 - (1 - e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}]^{\beta_2 - 1}$$

$$= \alpha_1{}^{n} \alpha_2{}^{m} \beta_1{}^{n} \beta_2{}^{m} \theta^{n+m} \lambda^{(n+m)\theta} \times \prod_{i=0}^{n} x_i^{-\theta-1} [1 - e^{-(\frac{\lambda}{x_i})^{\theta}}]^{\alpha_1 - 1} [1 - (1 - e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}]^{\beta_1 - 1}$$

$$\times e^{-\sum_{i=1}^{n} (\frac{\lambda}{x_i})^{\theta} - \sum_{j=1}^{m} (\frac{\lambda}{y_j})^{\theta}} \prod_{j=0}^{m} y_j^{-\theta-1} [1 - e^{-(\frac{\lambda}{y_j})^{\theta}}]^{\alpha_2 - 1} [1 - (1 - e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}]^{\beta_2 - 1}.$$

(3.16)

The logarithm of the joint likelihood function may be written as:

$$\ell(x, y, \underline{\phi}) = log\ L(x, y; \underline{\phi})$$

$$= n \log \alpha_1 + n \log \beta_1 + m \log \alpha_2 + m \log \beta_2 + (n+m) \log \theta + \theta(n+m) \log \lambda$$

$$-(\theta + 1)[\sum_{i=1}^{n} \log x_i + \sum_{j=1}^{m} \log y_j] - \sum_{i=1}^{n} (\frac{\lambda}{x_i})^{\theta} - \sum_{j=1}^{m} (\frac{\lambda}{y_j})^{\theta}$$

$$+(\beta_1 - 1) \sum_{i=1}^{n} \log[1 - (1 - e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}] + (\beta_2 - 1) \sum_{j=1}^{m} \log[1 - (1 - e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}].$$

(3.17)

The MLE can be obtained by differentiating Eq. (3.17)with respect to $\alpha_1, \beta_1, \beta_2, \alpha_2, \lambda, \theta$, and solving the following equations:

$$\frac{\partial \ell}{\partial \beta_1} = \frac{n}{\beta_1} + \sum_{i=1}^{n} \log[1-(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}] = 0, \tag{3.18}$$

$$\frac{\partial \ell}{\partial \beta_2} = \frac{m}{\beta_2} + \sum_{j=1}^{m} \log[1-(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}] = 0, \tag{3.19}$$

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n} \log[1-e^{-(\frac{\lambda}{x_i})^{\theta}}] + (\beta_1 - 1)\sum_{i=1}^{n} \frac{-(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1} \times \log(1-e^{-(\frac{\lambda}{x_i})^{\theta}})}{[1-(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}]} = 0, \tag{3.20}$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{m}{\alpha_2} + \sum_{j=1}^{m} \log[1-e^{-(\frac{\lambda}{y_j})^{\theta}}] + (\beta_2 - 1)\sum_{j=1}^{m} \frac{-(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2} \times \log(1-e^{-(\frac{\lambda}{y_j})^{\theta}})}{[1-(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}]} = 0, \tag{3.21}$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{\theta(n+m)}{\lambda} - \sum_{i=1}^{n} \theta(\frac{1}{x_i})(\frac{\lambda}{x_i})^{\theta-1} - \sum_{j=1}^{m} \theta(\frac{1}{y_j})(\frac{\lambda}{y_j})^{\theta-1} + \theta(\alpha_1 - 1)\sum_{i=1}^{n} \frac{e^{-(\frac{\lambda}{x_i})^{\theta}}(\frac{1}{x_i})(\frac{\lambda}{x_i})^{\theta-1}}{[1-e^{-(\frac{\lambda}{x_i})^{\theta}}]}$$

$$+ \theta(\alpha_2 - 1)\sum_{j=1}^{m} \frac{e^{-(\frac{\lambda}{y_j})^{\theta}}(\frac{1}{y_j})(\frac{\lambda}{y_j})^{\theta-1}}{[1-e^{-(\frac{\lambda}{y_j})^{\theta}}]} + \theta\alpha_1(\beta_1 - 1)\sum_{i=1}^{n} \frac{e^{-(\frac{\lambda}{x_i})^{\theta}}(\frac{1}{x_i})(\frac{\lambda}{x_i})^{\theta-1}}{[1-(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}]}$$

$$+ \theta\alpha_2(\beta_2 - 1)\sum_{j=1}^{m} \frac{e^{-(\frac{\lambda}{y_j})^{\theta}}(\frac{1}{y_j})(\frac{\lambda}{y_j})^{\theta-1}}{[1-(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}]}, \tag{3.22}$$

and

$$\frac{\partial \ell}{\partial \theta}=\frac{(n+m)}{\theta}+(n+m)\log\lambda-\sum_{i=1}^{n}\log x_i-\sum_{j=1}^{m}\log y_j-\sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\theta}\log(\frac{\lambda}{x_i})$$

$$-\sum_{j=1}^{m}(\frac{\lambda}{y_j})^{\theta}\log(\frac{\lambda}{y_j})+(\alpha_1-1)\sum_{i=1}^{n}\frac{e^{-(\frac{\lambda}{x_i})^{\theta}}(\frac{\lambda}{x_i})^{\theta}\log(\frac{\lambda}{x_i})}{(1-e^{-(\frac{\lambda}{x_i})^{\theta}})}$$

$$+(\alpha_2-1)\sum_{j=1}^{m}e^{-(\frac{\lambda}{y_j})^{\theta}}\frac{(\frac{\lambda}{y_j})^{\theta}\log(\frac{\lambda}{y_j})}{(1-e^{-(\frac{\lambda}{y_j})^{\theta}})}+\alpha_1(\beta_1-1)\sum_{i=1}^{n}\frac{[1-e^{-(\frac{\lambda}{x_i})^{\theta}}]^{\alpha_1-1}e^{-(\frac{\lambda}{x_i})^{\theta}}(\frac{\lambda}{x_i})^{\theta}\log(\frac{\lambda}{x_i})}{[1-(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}]}.$$

(3.23)

Obtaining a closed form expressions for the MLEs of the unknown parameters $\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda,\theta$ are not possible, so these nonlinear equations are solved numerically using iterative process as Newton Raphson to get $\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1,\hat{\beta}_2,\hat{\lambda},\hat{\theta}$.

Then the MLE of R can be obtained using the invariance property of the maximum likelihood estimator from Eq.(3.12) as following:

$$\hat{R}=\hat{\alpha}_1\hat{\beta}_1\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{\infty}(-1)^{(j_1+j_2+j_3+j_4)}\binom{\hat{\beta}_1-1}{j_1}\binom{\hat{\alpha}_1(j_1+1)-1}{j_2}\binom{\hat{\beta}_2}{j_3}\binom{\hat{\alpha}_2 j_3}{j_4}\frac{1}{(j_2+j_4+1)}.$$

(3.24)

Similarly, We can calculate the MLE of reliability for multi-component stress-strength model from Eq(3.15)by:

$$\hat{R}_{s,k}=\sum_{i=s}^{k}\sum_{j_1=0}^{i}\sum_{j_2=0}^{\hat{\beta}_1(j_1+k-i)}\sum_{j_3=0}^{\hat{\beta}_2-1}\binom{k}{i}\binom{i}{j_1}\binom{\hat{\beta}_1(j_1+k-i)}{j_2}\binom{\hat{\beta}_2-1}{j_3}(-1)^{j_1+j_2+j_3}\frac{\hat{\alpha}_2\hat{\beta}_2}{\hat{\alpha}_1 j_2+\hat{\alpha}_2(j_3+1)}.$$

(3.25)

3.7 Bayesian Estimation of R

In this section we provide the Bayes estimate of R where $\underline{\phi}=(\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda,\theta)$ are unknown parameters and all of these parameters having independent gamma prior distribution as following:

$$\pi(\lambda)=\frac{b_1^{a_1}}{\Gamma a_1}\lambda^{a_1-1}e^{-b_1\lambda},$$

$$\pi(\theta)=\frac{b_2^{a_2}}{\Gamma a_2}\theta^{a_2-1}e^{-b_2\theta},$$

$$\pi(\alpha_1)=\frac{b_3^{a_3}}{\Gamma a_3}\alpha_1^{a_3-1}e^{-b_3\alpha_1},$$

$$\pi(\alpha_2)=\frac{b_4^{a_4}}{\Gamma a_4}\alpha_2^{a_4-1}e^{-b_4\alpha_2},$$

$$\pi(\beta_1)=\frac{b_5^{a_5}}{\Gamma a_5}\beta_1^{a_5-1}e^{-b_5\beta_1},$$

and

$$\pi(\beta_2)=\frac{b_6^{a_6}}{\Gamma a_6}\beta_2^{a_6-1}e^{-b_6\beta_2}.$$

The joint posterior pdf is defined as

$$g(\underline{\phi}/data)=\frac{L(x,y/\alpha_1,\beta_1,\lambda,\theta,\alpha_2,\beta_2)\pi(\lambda)\pi(\theta)\pi(\alpha_1)\pi(\alpha_2)\pi(\beta_1)\pi(\beta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(x,y/\alpha_1,\beta_1,\lambda,\theta,\alpha_2,\beta_2)\pi(\lambda)\pi(\theta)\pi(\alpha_1)\pi(\alpha_2)\pi(\beta_1)\pi(\beta_2)d\underline{\phi}}.$$

Where $L(x,y/\alpha_1,\beta_1,\lambda,\theta,\alpha_2,\beta_2)$ is defined in Eq.(3.16), then the joint posterior function can be written as :

$$g(\underline{\phi}/data) \propto \alpha_1^{\ n}\alpha_2^{\ m}\beta_1^{\ n}\beta_2^{\ m}\theta^{n+m}\lambda^{(n+m)\theta}\prod_{i=1}^{n} x_i^{\ -\theta-1}(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1-1}[1-(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}]^{\beta_1-1}$$

$$\times e^{-\sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\theta}-\sum_{j=1}^{m}(\frac{\lambda}{y_j})^{\theta}} \times \prod_{j=1}^{m} y_j^{\ -\theta-1}(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2-1}[1-(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}]^{\beta_2-1}$$

$$\times \lambda^{a_1-1}e^{-b_1\lambda}\theta^{a_2-1}e^{-b_2\theta}\alpha_1^{\ a_3-1}e^{-b_3\alpha_1}\alpha_2^{\ a_4-1}e^{-b_4\alpha_2}\beta_1^{\ a_5-1}e^{-b_5\beta_1}\beta_2^{\ a_6-1}e^{-b_6\beta_2}.$$

(3.26)

Therefore, the Bayes estimate of reliability, say $\hat{R}_B$ under the squared error loss function isgiven by:

$$\hat{R}_B = \int_0^{\infty} R\ g(\underline{\phi}/data)d\underline{\phi}. \qquad (3.27)$$

It is impossible to compute the bayes estimate of R analytically using Eq.(3.27), therefore instead, we propose to approximate it by a Monto Carlo method to obtain this integration. Importance sampling technique was used for solving this problem.

3.7.1 Bayes estimate of R using Importance Sampling technique

It is so difficult to generate samples directly from the posterior function in Eq.(3.26), so we divided it to individuals function which easy to generate sample from them.

So we can rewrite Eq.(3.26)as following:

$$g(\underline{\phi}/data) = g_1(\theta/data)g_2(\lambda/\theta,data)g_3(\alpha_1/\lambda,\theta,data)g_4(\alpha_2/\lambda,\theta,data)$$
$$\times g_5(\beta_1/data)g_6(\beta_2/data)h(\underline{\phi}/data)$$

where

$$g_1(\theta/data) \propto Gamma\left(b_2 + \sum_{i=1}^{n} lnx_i + \sum_{j=1}^{m} lny_j, a_2 + n + m\right), \tag{3.28}$$

$$g_2(\lambda/\theta,data) \propto Gamma\left(b_1, a_1 + (n+m)\theta\right), \tag{3.29}$$

$$g_3(\alpha_1/\lambda,\theta,data) \propto Gamma\left(b_3 - \sum_{i=1}^{n} ln(1 - e^{-(\frac{\lambda}{x_i})^{\theta}}), a_3 + n\right), \tag{3.30}$$

$$g_4(\alpha_2/\lambda,\theta,data) \propto Gamma\left(b_4 - \sum_{j=1}^{m} ln(1 - e^{-(\frac{\lambda}{y_j})^{\theta}}), a_4 + m\right), \tag{3.31}$$

$$g_5(\beta_1/data) \propto Gamma\left(b_5, a_5 + n\right), \tag{3.32}$$

$$g_6(\beta_2/data) \propto Gamma\left(b_6, a_6 + m\right), \tag{3.33}$$

and

$$h(\underline{\phi}/data) = e^{-\sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\theta} - \sum_{j=1}^{m}(\frac{\lambda}{y_j})^{\theta}} \prod_{i=1}^{n}\left[1-(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}\right]^{\beta_1 - 1} \prod_{j=1}^{m}\left[1-(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}\right]^{\beta_2 - 1}$$

$$\times \frac{\Gamma(a_1 + (n+m)\theta) e^{-\sum_{i=1}^{n} ln(1-e^{-(\frac{\lambda}{x_i})^{\theta}})} e^{-\sum_{j=1}^{m} ln(1-e^{-(\frac{\lambda}{y_j})^{\theta}})}}{b_1^{(a_1+n+m)\theta}[b_3 - \sum_{i=1}^{n} ln(1-e^{-(\frac{\lambda}{x_i})^{\theta}})]^{a_3+n}[b_4 - \sum_{j=1}^{m} ln(1-e^{-(\frac{\lambda}{y_j})^{\theta}})]^{a_4+m}}. \tag{3.34}$$

It is clear that

$g_1(\theta/data)$, $g_2(\lambda/\theta, data)$, $g_3(\alpha_1/\lambda, \theta, data)$, $g_4(\alpha_2/\lambda, \theta, data)$,

$g_5(\beta_1/data)$, and $g_6(\beta_2/data)$ follow a gamma distribution, so it is quite simple to generate samples from them. The following algorithm will be used assuming that $a_1, \cdots, a_6$ and $b_1, \cdots, b_6$ are known apriori, and assuming initial values for $\lambda, \theta, \alpha_1, \alpha_2, \beta_1, \beta_2$.

Importance Sampling Algorithm:

- Step1: Generate θ_1 from $g_1(./data)$.
- Step2: Generate λ_1 from $g_2(./\theta, data)$.
- Step3: Generate α_{11} from $g_3(./\lambda, \theta, data)$, α_{21} from $g_4(./\lambda, \theta, data)$.
- Step4: Generate β_{11} from $g_5(./data)$, β_{21} from $g_6(./data)$.
- Step5: Repeat this procedure N times to obtain

$$(\theta_1, \lambda_1, \alpha_{11}, \alpha_{21}, \beta_{11}, \beta_{21}), \cdots, (\theta_N, \lambda_N, \alpha_{1N}, \alpha_{2N}, \beta_{1N}, \beta_{2N}).$$

- Step6: An approximate Bayes estimate of R under a squared error loss function can be obtained as

$$\hat{R}_B = \frac{\frac{1}{N}\sum_{i=1}^{N} R_i h(\theta_i, \lambda_i, \alpha_{1i}, \alpha_{2i}, \beta_{1i}, \beta_{2i}/data)}{\frac{1}{N}\sum_{i=1}^{N} h(\theta_i, \lambda_i, \alpha_{1i}, \alpha_{2i}, \beta_{1i}, \beta_{2i}/data)},$$

where

$$R_i = R(\theta_i, \lambda_i, \alpha_{1i}, \alpha_{2i}, \beta_{1i}, \beta_{2i}),$$

as defined in Eq. (3.12) for $i = 1, \cdots, N$.

Using the same technique, We can obtained the bayesian estimation of reliability for multi-component stress-strength model by replacing R by $R_{s,k}$ given in Eq. (3.15).

3.8 Numerical Study

In this section, we mainly present some simulation experiments to see the behavior of the proposed methods for various sample sizes and for parameter values $\alpha_1 = 0.75, \alpha_2 = 1.5, \beta_1 = 3.5, \beta_2 = 2.2, \lambda = 1.008, \theta = 0.61$, so that the true reliability value is 0.847751. We compared the performances of the MLEs and the Bayes estimates with respect to the squared error loss function in terms of biases and mean squares errors (MSEs).

We have taken sample sizes namely (n, m)= (5, 5), (10, 10), (20, 20), (30, 30).

For bayesian estimation, we used importance sampling technique under the informative gamma priors. For choosing a suitable hyper-parameters, the experimenters can incorporate their prior guess in terms of location and precision for the parameter of interest. The gamma distribution has mean = a/b, and varience = a/b^2. We assume a small value of prior varience (0.005), and taken the mean equal to the parameter of interest. For each parameter priors we solve the two equations of the mean and the varience, we obtain the following values of hyper-parameters :

$a_1 = 201.6,\ a_2 = 76.25,\ a_3 = 107.14,\ a_4 = 500,\ a_5 = 2500,\ a_6 = 956.5.$

and $b_1 = 200, b_2 = 125, b_3 = 142.857, b_4 = 333.333, b_5 = 714.286, b_6 = 434.783$

For the all mentioned sample sizes, we obtained the average estimate, bias and the mean squared errors of the MLE and Bayesian estimation of the stress-strength reliability R which given in Table (3.1).

Table 3.1: Average estimate, Bias and MSE of R using different estimators.

Estimators	MLE			Bayesian		
(n,m)	$\hat{R}$	Bias	MSE	$\hat{R}_B$	Bias	MSE
(5,5)	0.90616	0.05841	0.01502	0.88337	0.0316	0.00425
(10,10)	0.89713	0.04938	0.01116	0.87597	0.02822	0.00124
(20,20)	0.89001	0.04226	0.00612	0.86825	0.0205	0.00096
(30,30)	0.87338	0.02563	0.00379	0.86002	0.01227	0.0007

3.9 Real Data Analysis

In this section, we present a data analysis of the strength data introduced by Badar and Priest (1982). The data stand for the strength data measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. For illustrative purpose, we consider the data sets consisting the single fibers of 20 mm (Data Set 1) and 10 mm in gauge lengths (Data Set 2), with sample sizes 69 and 63 respectively. Data sets are provided below:

Data set 1:(strength measurements)

.312, .314, .479, .552, .7, .803, .861, .865, .944, .958, .966, .997, 1.006, 1.021, 1.055, 1.063, 1.098, 1.14, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585.

Data set 2:(stress measurements)

.101, .332, .403, .428, .457, .550, .561, .596, .597, .645, .654, .674, .718, .722, .725, .732, .775, .814, .816, .818 ,.824, .859, .875, .938, .940, 1.056, 1.117, 1.128, 1.137, 1.137, 1.177, 1.196, 1.230, 1.325, 1.339, 1.345, 1.420, 1.423, 1.435, 1.443, 1.464, 1.472, 1.494, 1.532, 1.546, 1.577, 1.608, 1.635, 1.693, 1.701, 1.737, 1.754, 1.762, 1.828, 2.052, 2.071, 2.086, 2.171, 2.224, 2.227, 2.425, 2.595, 3.2.

We fit the two data sets separately with the exponentiated generalized inverse weibull distribution (EGIW). we provide the Kolmogorov-Smirnov (K-S), Anderson-Darling(A-D) and Cramér-von Mises (C-V) goodness-of-fit tests in Table(3.2). Obviously, the (EGIW) model fits well to Data Set 1 and Data Set 2.

The MLE and Bayesian estimates of R for the real data are provided in Table (3.3).

Table 3.2: P-value of different goodness-of-fit tests for data set 1, 2.

	K-S	A-D	C-V
data set 1.	0.231248	0.143961	0.152425
data set 2.	0.192997	0.126852	0.213019

Table 3.3: Maximum likelihood,Bayesian estimates of the parameters and R.

	α_1	α_2	β_1	β_2	λ	θ	R
MLE	2.7192	1.9639	4.4707	2.0057	0.9511	1.0789	0.55826
Bayes	1.1070	1.5513	3.6196	2.1963	1.5344	1.06724	0.7493

In case of multi-component stress-strength model, the maximum likelihood and Bayes estimates of the stress-strength reliability based on the real data sets, are presented in Table (3.4) for different values of (s, k).

Table 3.4: The Maximum likelihood and Bayesian estimates of $R_{s,k}$.

$R_{s,k}$	MLE	Bayes
(1,3)	0.73573	0.82293
(1,5)	0.83869	0.91084
(2,4)	0.54955	0.70667
(3,3)	0.16096	0.34609
(3,5)	0.42262	0.6608

3.10 Conclusions

In this chapter we presented two methods for estimating $R = Pr(Y < X)$ when X and Y both follow exponentiated generalized inverse weibull distribution with different parameters. We investigated Maximum likelihood and Bayesian estimation methods of R and their performances are examined by simulation study.

We have computed the Bayes estimate of R based on the independent gamma priors and using squared error loss function. Since the Bayes estimate cannot be obtained in explicit form, we have used the importance sampling technique to compute the Bayes estimate.

From the simulation results given earlier in **Table** 3.1, we observed that:

• For all the methods, as the sample size increases the biases and the mean squared errors decrease.

• The performance of the Bayes estimators is better than maximum likelihood for all different sample sizes.

• Maximum likelihood results are improving and become closer to Bayesian results as sample size increased.

Real data analysis has been performed for illustrative purposes. We introduced the MLE, and Bayesian estimation of multi-component stress-strength reliability using the real data study. From the results given in Table 3.4, we notice that, for fixed k, as s increases then the value of $R_{s,k}$ decreases, also for fixed s, as k increases then the value of $R_{s,k}$ increases.

FUTURE WORK

- Study the stress-strength reliability estimation for another distributions.
- Study the stress-strength reliability estimation when the stress follow a distribution and the strength follow another distribution.
- Using another methods of estimation(Moment method, Bootstrap confidence interval) and another types of loss functions, and approximation method(Lindely Approximation) for Bayesian method.
- Study the stress-strength reliability estimation based on censored samples**.**

References

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الملخص العربي

يعد نموذج صلاحية القوة والاجهاد من اهم النماذج التي تقيس صلاحية منتج أو أي نظام مكون من عدة وحدات في كثير من التطبيقات الهندسيه والطبية والصناعية.

فإذا فرضنا أن قوة تحمل أي نظام أو وحدة بمفردها يرمز له Xوالإجهاد الواقع عليه يرمز له Y، فأن هذا النظام(أو الوحده)يظل يعمل ويؤدي وظيفته، مادام أن قوة تحمله Xأكبر من الإجهاد المؤثر عليهY،وهذة هي الفكرة الاساسيه التي يعتمد عليها نموذج صلاحية القوة و الاجهاد.لذلك فإن دالة صلاحية القوة و الإجهاد R توصف رياضيا بأنها دالة احتمال ان Y اقل من X "$R = \Pr(Y < X)$".

الهدف الرئيسي من هذه الرساله هو دراسة الإستدلال الإحصائيلمعالم صلاحية القوة والإجهاد لتوزيعات إحصائيه مختلفة وذلكبإستخدام طرق تقدير مختلفه. حيث نعتبر القوة والإجهاد كمتغيرين عشوائيين يتبعان نفس التوزيع الإحصائي مع اختلاف معالم كل منهما.لقد قمنابحساب معالم صلاحية القوة والاجهاد باستخدام طريقة الإمكان الإعظم وطريقةبايز، كما تم إيجاد فترات الثقه التقريبيه للصلاحيه. في طريقة بايز استخدمنا دوال مختلفة للخسارة (متماثلة وغير متماثلة). وكمثال توضيحي استخدمنا أسلوب المحاكاة للمقارنه بين طرق التقدير المستخدمه من حيث متوسط مربع الخطأ(MSE). كما تم استخدام بيانات حقيقيه وتقدير قيمة الصلاحية لها.

تحتوي هذه الرسالة على ثلاثة أبواب فيما يلي عرضموجزعن كل باب:

الباب الأول:

وهو عبارة عن مقدمه يتم فيها عرض بعض التعريفات والمفاهيم الأساسيه المستخدمه في الرساله. يناقش هذا الباب بشكل عام موضوعات الصلاحية وصلاحية القوة والإجهاد وأيضا نموذج القوة والإجهاد للأنظمه متعددة الوحدات. كما يعرض أيضا بعض الطرق المستخدمةلتقدير معالم التوزيعات الإحصائية ومعالم صلاحية القوة والإجهاد. و يناقش أيضابعض طرق(Markov Chain Monte Carlo) لإيجاد طريقة مناسبه فيحسابات تقدير بايز.كذلك تم الإشارة الي بعض التوزيعات الإحصائيه التي تم

إستخدامها في الرساله، بالإضافه الي عرض تاريخي لموضوع الرساله.

الباب الثاني:

في هذا الباب تم إيجاد تقدير معالم صلاحية القوة والإجهاد عندما يتبع دالتي القوة والإجهاد توزيع(Quasi Lindley) مع اختلاف معالم كل منهما. حيث تم إستخدام طريقة الإمكان الأعظم وطريقةبايزاعتمادا على دوال خسارة متماثله وأخري غير متماثله لإيجاد معالم الصلاحية. كذلك تم حساب فترة الثقه التقريبيه للصلاحيه. ثم استعرضنا نتائج المحاكاة كمثال توضيحي للمقارنه بين طرق التقدير المستخدمة من حيث متوسط مربع الخطأ(MSE)، كما تم تقدير صلاحية القوة والإجهاد لبيانات حقيقيه مقترحة.الجدير بالذكر أن نتائج هذا الباب تم نشرها في المجلة الدولية:

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الباب الثالث:

يناقش هذا الباب كيفية تقدير معالم صلاحية القوة والإجهاد عندما يتبع دالتي القوة والاجهاد توزيع (Exponentiated Generalized Inverse Weibull) مع إختلاف معالم كل منهما. حيث تم إستخدام طريقة الإمكان الأعظم وطريقة بايز لإيجاد معالم الصلاحية للنظامذات الوحدة الواحدة وكذلك للنظام متعدد الوحدات. ثم استعرضنا نتائج المحاكاة للمقارنة بين طرق التقدير المستخدمة من حيث متوسط مربع الخطأ(MSE)،كذلك تم إقتراح بيانات حقيقيه وتقدير صلاحية القوة والإجهاد لهذة البيانات. الجدير بالذكر أن نتائج هذا الباب تم نشرها في المجله الدوليه:

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شكر وتقدير

بداية أشكر الله من فضل وتكرم، وأعطي وأنعم، ووفق ويسر،خالقي ورازقي وولي نعمتي ربي ورب كل شيء، فالحمد لله الذ ي تتم بنعمته الصالحات، لقد وفقني الله سبحانه وتعالي بفضله وجوده ومنه وكرمه في إنجاز هذة الرساله، فالحمد لله ع الدوام وله الشكر على التمام.

والصلاة والسلام علي سيد الأنام، وحبيب الرحمن سيدنا محمد " صلي الله عليه وسلم" معلم البشرية الذي حثنا علىطلب العلم فصلاةً وسلاماً عليه ننال بهما في الدنيا عزة وكرامه، وفي الاخرة صحبه وشفاعه.

لا يسعني إلا أن أتقدم بعميق الشكر، والإمتنان لهيئة الإشراف وهم:

الأستاذ الدكتور/ مصطفي محمد محي الدين – أستاذ الإحصاء الرياضي بجامعة الأزهر الذي أحاطني بالاهتمام من خلال اقتراحه لموضوعات البحث وإرشاده المستمر وصبره وعطائه اللامحدود، فلقد كان دائما محفزا لي لإنجاز هذا البحث وتعلمت منه النقد البناء، وأعطاني الكثير من وقته الثمين والكثير من التوجيهات القيمه.

الدكتورة/ صديقه أحمد عبدالله– أستاذ مساعد الإحصاء الرياضي بكلية الهندسة ببنها– جامعة بنها، لتشجيعها ومشاركتها في هذه الرساله.

الدكتور/ عمرو فؤاد احمد صادق– أستاذ مساعد الإحصاء الرياضي بجامعة الازهر لما قدمه من دعم متواصل وتوجيهات أثناء إعداد هذه الرسالة، وقد بذل أقصي ما لديه من جهد ووقت لنجاح هذا العمل من خلال المراجعة القيمه. كما أدين له بالكثير من الفضل في تعلم البرامج التي استخدمتها أثناء اعداد الرساله.

وأتقدم أيضا بجزيل الشكر لأساتذتي ورفقاء البحث العلمي بمدرسة الإحصاء الرياضي برئاسة **الأستاذ الدكتور/ مصطفي محمد محي الدين،** على تشجيعهم واقتراحاتهم، وأخص بالذكر **الدكتور/ مجدي ناجي أحمد** بجامعة الفيوم، و**الدكتور/عبدالرحيم محمد عبدالرحيم** بتربيةعين شمس علي مساعدتهم في إنجاز نتائج وحسابات هذا البحث فلم يبخلوا عني بأي معلومة.

كما أعجز أن أقدم تقديرا مناسبا لأبي وأمي وزوجي وأولادي (مالك وحمزة)، وأخوتي وأخواتي لتشجيعهم ودعمهم المستمر لإنجاز هذه الرساله، فلولا مساندتهم ما كان هذا العمل. شكرا جزيلا لكل ما قدمتموه لي .

وأخيرا أتقدم بالشكر لأساتذتي وزملائي بكليتي الهندسة ببنها وشبرا الذين مدوا يد العون لمساعدتي أثناء إعداد هذه الرساله.

إهداء

إلى من آزراني في محنتي وتحمل مشقتي، إلي رفيق دربي في هذة الحياه،إلى زوجي العزيز.

إلى من علمني الصبر والنجاح، إلى مدرستي الاولي في الحياة، إلى والدي الغالي على قلبي أطال الله في عمره.

إلى والدتي الغالية التي لم تألُ جهداً في تربيتي ومساعدتيلأقدم هذا العمل.

إلى فلذات كبدي،إلى أبنائي الأعزاء (مالك وحمزة).

إلى أخوتي وأخواتي وإلى أسرتي جميعا.........

بكل الحب والتقدير والامتنان، أهدي إليكم هذا العمل المتواضع.

داعيةً المولي –عز وجل– ان تكلل بالنجاح والقبول.

جامعة بنها

كلية الهندسة بشبرا

قسم الرياضيات والفيزيقا الهندسية

صلاحية القوة والإجهاد لبعض التوزيعات الإحصائية

رسالة مقدمة كجزء من متطلبات الحصول على درجة الماجستير في الرياضياتوالفيزيقا الهندسية (الرياضيات الهندسية)

مقدمة من

المهندسة/ شيماء حامد عبدالله المغاوري محمد

تحت اشراف

أ. د. مصطفي محمد محي الدين

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